PLATE AND LINE SEGMENT PROCESSES

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Abstract
Random processes of convex plates and line segments imbedded in $R^3$ are considered in this paper, and the expected values of certain random variables associated with such processes are computed under a mean stationarity assumption, by resorting to some general formulas of integral geometry.

GEOMETRICAL PROBABILITY; INTEGRAL GEOMETRY; RANDOM SETS; CONVEX PLATES; LINE SEGMENTS; POISSON DISTRIBUTION

1. Introduction

Random processes of diverse geometric figures have been studied in several papers in view of their applications. Coleman (1974), Parker and Cowan (1976) and more recently Berman (1977) are some of the authors who have solved several problems regarding this kind of process.

In the first part of this paper we consider a random process of convex plates in $R^3$, computing the expected values of certain random variables associated with such a process under a fairly general assumption of stationarity which is obviously satisfied in the case of the strongly stationary Poisson process. The following section is devoted to line segment processes of the type considered by Parker and Cowan (1976). In this connection, we show how the formulas they state for the three-dimensional case, as well as those they derive for the plane, can be obtained from the now classical results of integral geometry as exposed, for example, in the work of Santaló (1936), (1976).

In the last section, we consider a mixed process of plates and of line segments. By 'plates' we shall always understand convex plates.

2. Plate processes

Let us consider in $R^3$ a random process of oriented plates depending on a non-negative random parameter $\rho$, in such a way that any two plates which correspond to the same value of $\rho$ are congruent.

Suppose also that each plate of the family contains a 'distinguished' point $H$, such that any two congruent plates, if carried to coincide, the distinguished points coincide as well.

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We shall assume that our plate process can be decomposed into four mutually independent processes, namely:

(i) The distinguished points $H$, which form a point process with the property that the expected value of the random variable $H(A) =$ the number of points $H$ within the Borel set $A \subset \mathbb{R}^3$ is invariant under translations (in keeping with usage, we call this property 'mean stationarity').

(ii) The unit normal vectors $u$ to each plate, which we shall assume to be mutually independent and uniformly distributed on the unit spherical surface $S^2$, with density $du/4\pi$, where $du$ is the area element of $S^2$.

(iii) The angles $\varphi$ formed by an oriented segment in each plate with an oriented segment in the plane containing the plate. We assume that these angles are mutually independent and uniformly distributed on the interval $[0,2\pi)$.

(iv) The values of $\rho$ for each plate of the process, which we shall assume to be mutually independent with a common distribution function $F(\rho)$ concentrated in $[0,\infty)$ and such that if $f = f(\rho)$ is the area and $s = s(\rho)$ the perimeter of a plate with parameter $\rho$, then

\[ E(f) = \int_0^\infty f(\rho)dF(\rho) \quad \text{and} \quad E(s) = \int_0^\infty s(\rho)dF(\rho) \]

are both finite.

Three further assumptions are:

(v) the points $H$ can be labelled according to their distance to the origin — for two or more points at an equal distance, a systematic method for continuing the labelling is chosen;

(vi) the process of points $H$ is almost surely orderly, that is, $Pr(H(\{x\}) = 0 \text{ or } 1$ for every $x \in \mathbb{R}^3) = 1$;

(vii) the expected value of $H(A)$ is finite if $A$ is bounded.

We remark that this model formulation is completely analogous to the one given by Parker and Cowan (1976).

Since each plate $\mathcal{P}$ of the family can be identified with the ordered set $(H,u,\varphi,\rho)$, our plate process is equivalent to a point process in the product space

\[ Z = \mathbb{R}^3 \times S^2 \times [0,2\pi) \times [0,\infty) \]

which we call the associated point process (APP).

Since $EH(\cdot)$ is a measure on the Borel subsets of $\mathbb{R}^3$, our first assumption entails the equation

\[ EH(A) = \lambda m(A), \]

where $m$ stands for Lebesgue measure and $\lambda$ is a constant called 'intensity' of the point process (i).

From now on, we shall denote by $A$ an arbitrary convex set in $\mathbb{R}^3$. 
In order to compute the expected value of the random variable \( N(A) \) = the number of plates of the process which intersect with \( A \), we consider on the \( \sigma \)-field of Borel subsets of \( Z \) the product measure

\[
d\mathcal{P} = \left( \frac{\lambda}{8 \pi^2} \right) d\mathcal{H}du \varphi dF(\rho),
\]

whose integral over any Borel set \( U \subset Z \) represents the average number of points of the \( \text{APP} \) within \( U \). By \( d\mathcal{H} = dx dy dz \), we denote the volume element of \( \mathbb{R}^3 \).

The measure (4) can be written in the form

\[
d\mathcal{P} = \left( \frac{\lambda}{8 \pi^2} \right) dK dF(\rho),
\]

where \( dK = d\mathcal{H}du \varphi \) is the kinematic density of integral geometry (Santaló (1976), Chapter 15). Hence, we may compute the expected value of \( N(A) \) as follows:

\[
EN(A) = \int d\mathcal{P} = \frac{\lambda}{8 \pi^2} \int_0^\infty dF(\rho) \int_{\mathcal{P} \cap \Lambda} dK,
\]

where the last integral extends over the region of all plates \( \mathcal{P} \) with parameter \( \rho \) that intersect with \( A \).

It is known (Santaló (1976), Formulas (13.66) and (15.63) for \( n = 3 \)) that the value of the last integral is \( 8 \pi^2 V_A + \pi^2 F_A s + 4 \pi M_A f \), where \( V_A \), \( F_A \) and \( M_A \) represent the volume, the surface area and the mean curvature of \( A \). Therefore

\[
EN(A) = \frac{\lambda}{8 \pi^2} \int_0^\infty \{8 \pi^2 V_A + \pi^2 F_A s(\rho) + 4 \pi M_A f(\rho)\} dF(\rho)
\]

\[
= \lambda V_A + \frac{1}{8} \lambda F_A E(s) + (\lambda M_A / 2 \pi) E(f).
\]

We shall denote by \( \alpha = \alpha(H, u, \varphi, \rho) \) the area of the intersection of the plate \( \mathcal{P} \) with \( A \).

If we wish to compute the expected value of the random variable \( S(A) = \text{total area within } A \text{ of all plates of the process which intersect with } A \), we have

\[
ES(A) = \int \alpha d\mathcal{P} = \frac{\lambda}{8 \pi^2} \int_0^\infty dF(\rho) \int \alpha dK.
\]

On the other hand, the formula (15.20) of Santaló (1976) in the particular case under consideration, gives us the value

\[
\int \alpha dK = 8 \pi^2 V_A f(\rho),
\]

where \( V_A = m(A) \) is, as before, the volume of \( A \). Hence,
For our next computation we shall take use of the following hypothesis:
(viii) if $A$ and $B$ are disjoint Borel sets in $\mathbb{R}^3$, $E[H(A)H(B)] = EH(A) \cdot EH(B)$. The need for this assumption is carefully discussed in Parker and Cowan (1976).

For each pair $\mathcal{P}_i$ and $\mathcal{P}_j$ of different plates, we shall denote by $l(A \cap \mathcal{P}_i \cap \mathcal{P}_j)$ the length of the intersection within parentheses.

In order to calculate the expected value of the random variable $L(A) =$ total length within $A$ of the intersection of each pair of different plates of the process, we consider on the cartesian product $Z^2 = Z \times Z$ the product measure $d\mathcal{P}_1 d\mathcal{P}_2$, where $d\mathcal{P}_1$ and $d\mathcal{P}_2$ denote the measure (5).

The diagonal of $Z^2$ being a set of measure zero, we have

\begin{equation}
EL(A) = \frac{1}{2} \int l(A \cap \mathcal{P}_i \cap \mathcal{P}_j) d\mathcal{P}_1 d\mathcal{P}_2.
\end{equation}

where $\mathcal{P}_i$ is the plate corresponding to the point $(H_i, u_i, \varphi_i, \rho_i)$, $i = 1, 2$.

Using the factorization
\[d\mathcal{P}_i = \frac{\lambda}{8\pi^2} dK_i dF(\rho_i),\]
where $dK_i = dH_i du_i d\varphi_i$ is the kinematic density for $\mathcal{P}_i$, we can represent the right-hand member of (8) in the form
\[
\frac{\lambda^2}{128\pi^4} \int_0^\infty dF(\rho_2) \int_0^\infty dF(\rho_1) \int l(A \cap \mathcal{P}_i \cap \mathcal{P}_j) dK_i dK_2.
\]

On the other hand, the formula (15.22) of Santaló (1976) gives for the last integral the value $16\pi^3 V_\lambda f(\rho_i)f(\rho_j)$. Hence, it follows that

\begin{equation}
EL(A) = \frac{1}{8} \pi \lambda^2 V_\lambda E(f)^2.
\end{equation}

Next, we consider the random variable $T(A) =$ the number of intersections within $A$ of every three different plates of the process.

To this end, we introduce on the cartesian product $Z^3 = Z \times Z \times Z$ the product measure $d\mathcal{P}_1 d\mathcal{P}_2 d\mathcal{P}_3$, where each $d\mathcal{P}_i$ denotes the measure (5) and we assume:
(ix) if $A, B$, and $C$ are disjoint Borel sets in $\mathbb{R}^3$, then $E[H(A)H(B)H(C)] = EH(A)EH(B)EH(C)$. Under this assumption, the expected value of $T(A)$ is given by

\begin{equation}
ET(A) = \frac{1}{6} \int_{A \cap \mathcal{P}_i \cap \mathcal{P}_j \cap \mathcal{P}_k} d\mathcal{P}_1 d\mathcal{P}_2 d\mathcal{P}_3,
\end{equation}

where $\mathcal{P}_i$ corresponds to the point $(H_i, u_i, \varphi_i, \rho_i)$, $i = 1, 2, 3$. 
By factorizing each measure \(d\Phi_i\), through the corresponding kinematic density \(dK_i\), we get for the right-hand member of (10) the value

\[
\frac{\lambda^3}{3072\pi^6} \int_0^\infty \int_0^\infty \int_0^\infty dF(\rho_1)dF(\rho_2)dF(\rho_3) \lambda^{\rho_1+\rho_2+\rho_3} dK_1 dK_2 dK_3.
\]

The value of the last integral can be obtained from the formula (15.22) of Santaló (1976). The result is \(64\pi^7 V_A f(\rho_1)f(\rho_2)f(\rho_3)\). Hence

(11) \[ET(A) = \frac{\pi}{48} \lambda^3 V_A E(f)^3.\]

In passing, we remark that since each member of Equations (7) and (11) defines a measure on the Borel sets \(A \subset R^3\), it follows that they hold true for any Borel set \(A\).

It may also be useful to remark that our assumptions (viii) and (ix) are both satisfied in the case of the Poisson process, whose definition is given below.

The Poisson process. If we suppose, in addition to (i) through (vii), that the point process (i) is a homogeneous Poisson process of intensity \(\lambda\), then the random variable \(N(A)\) has a Poisson distribution with the expectation given by (6).

Let us call \(D\) the distance from the origin to the nearest plate of the process. If we take for \(A\) the ball of radius \(r\) centered at the origin, then \(P\{D > r\} = P\{N(A) = 0\}\). But for this particular \(A\) we have

\[V_A = \frac{4\pi r^3}{3}, \quad F_A = 4\pi r^2, \quad M_A = 4\pi r,\]

so that the expected value of \(N(A)\) is the number

(12) \[\mu = \frac{4\pi \lambda r^3}{3} + \frac{\pi \lambda r^2}{2} E(s) + 2\lambda r E(f).\]

Hence, under the Poisson assumption,

\[P\{D > r\} = e^{-\mu}\]

with \(\mu \) given by (12). From this, we readily obtain for the probability density function of \(D\) the expression

\[\lambda e^{-\mu} \{4\pi r^3 + \pi r E(s) + 2E(f)\}.\]

Examples. (a) If all plates of the process are similar, we may distinguish in each one of them the center of homothety \(H\) and we can take for \(\rho\) the ratio of similarity of each plate with respect to a fixed plate of the family. With this choice of the random parameter, we have \(f(\rho) = f(\rho)^2\) and \(s(\rho) = s(\rho)^2\), where \(f = f(1)\) is the area and \(s = s(1)\) the perimeter of the fixed plate. In this example we have \(E(f) = f, E(\rho^2)\) and \(E(s) = s, E(\rho)\).
(b) If the distribution function of the random parameter $p$ is concentrated on a single point, then the plate process becomes a process of congruent plates.

3. Line segment processes

Let us consider in $\mathbb{R}^3$ a random process of oriented line segments with variable length. We shall assume that this process can be decomposed into three mutually independent processes, namely:

(i) The mid points $M$ of each segment, which form a mean stationary process of intensity $\sigma$, that is, if $M(A)$ denotes the number of mid points within the Borel set $A$, then $EM(A) = \sigma m(A)$.

(ii) The unit vectors $u$ in the direction of each segment, which we assume to be mutually independent and uniformly distributed over the unit spherical surface $S^2$, with density $du/4\pi$, where $du$ is the area element of $S^2$.

(iii) The lengths $l$ of each segment, which we suppose mutually independent with a common distribution $G(l)$ concentrated in $[0, \infty)$.

Except for the minor hypothesis of orientation, these are the basic assumptions of the model formulated by Parker and Cowan (1976).

Since each segment $s$ can be identified with the ordered set $(M, u, l)$, the segment process is equivalent to a point process in the product space $W = \mathbb{R}^3 \times S^2 \times [0, \infty)$, which we call, as before, the associated point process (APP).

If we wish to compute the expected value of the random variable $N(A) = \text{the number of segments of the process which intersect with the convex set } A$, we are led to consider on the $\sigma$-field of Borel subsets of $W$ the product measure

\begin{equation}
\mathcal{d} \mathcal{G} = \frac{\sigma}{4\pi} \, dM \, du \, dG(l),
\end{equation}

where $dM = dx \, dy \, dz$ represents the Lebesgue measure in $\mathbb{R}^3$.

To conform to the kinematic fundamental formula of integral geometry, we introduce on the Borel subsets of the product space $W_1 = W \times [0, 2\pi)$ the product measure

\begin{equation}
\mathcal{d} \mu = \mathcal{d} \mathcal{G} \, d\phi,
\end{equation}

where $d\phi$ is the Lebesgue measure on $[0, 2\pi)$.

The measure (14) can be written in the form

\begin{equation}
\mathcal{d} \mu = \frac{\sigma}{4\pi} \, dK \, dG(l),
\end{equation}

where $dK$ is the kinematic density of Santaló (1976), Chapter 15 for $n = 3$.

Hence, we have

\[2\pi EN(A) = \int_{A \cap \mathcal{G} \mathcal{H}} \mathcal{d} \mu = \frac{\sigma}{4\pi} \int_0^\infty dG(l) \int_{A \cap \mathcal{G} \mathcal{H}} dK.\]
It is known (Santaló (1976), Formulas (13.65) and (15.63) for \( n = 3 \)) that the value of the last integral is \( 8\pi^2 V_\Lambda + 2\pi^2 F_\Lambda l \). Therefore
\[
EN(A) = \sigma V_\Lambda + \frac{\sigma}{4} F_\Lambda E(l).
\]

As to the random variable \( L(A) = \text{total length within } A \text{ of all segments of the process which intersect with } A \), let us denote by \( t(A \cap \mathcal{S}) \) the length of the intersection within parentheses. Then
\[
\int_{\mathcal{S}} t(A \cap \mathcal{S}) d\varphi d\mathcal{S} = \frac{\sigma}{4\pi} \int_{\mathcal{S}} t(A \cap \mathcal{S}) dKdG(l)
\]
and by applying Fubini's theorem in the last relation, we get
\[
2\pi EL(A) = \frac{\sigma}{4\pi} \int_0^\infty dG(l) \int t(A \cap \mathcal{S}) dK.
\]

According to Formula (15.20) of Santaló (1976), the value of the last integral is \( 8\pi^2 V_\Lambda l \). Hence
\[
EL(A) = \sigma V_\Lambda E(l).
\]

For the same reasons as before, the last formula holds for any Borel set \( A \).

4. Mixed process

Let us consider in \( R^2 \) a mixed process consisting of two mutually independent processes of plates and of line segments of the types defined in the preceding sections.

Keeping the previous notations, we consider the product space \( W_1 \times Z \) endowed with the product measure
\[
d\mu d\mathcal{P} = d\varphi d\mathcal{S} d\mathcal{P} = \frac{\lambda\sigma}{32\pi^3} dG(l) dF(\rho) dK, dK_\rho,
\]
where \( dK_\rho \) and \( dK_\rho \) stand respectively for the kinematic density of \( \mathcal{S} \) and of \( \mathcal{P} \).

To compute the expected value of the random variable \( I(A) = \text{the number of segment-plate intersections within the convex set } A \), we have the formula
\[
EI(A) = \int_{A \cap \mathcal{P} \cap \mathcal{S} \neq \emptyset} d\mathcal{P} d\mathcal{S}.
\]

Therefore, by integrating \( d\mu d\mathcal{P} \) over the set \( A \cap \mathcal{P} \cap \mathcal{S} \neq \emptyset \), we get
\[
2\pi EI(A) = \frac{\lambda\sigma}{32\pi^3} \int_0^\infty \int_0^\infty dG(l) dF(\rho) \int_{A \cap \mathcal{P} \cap \mathcal{S} \neq \emptyset} dK, dK_\rho.
\]

On the other hand, by virtue of Formula (15.22) of Santaló (1976), the value of
the last integral is $32\pi^* V alf$. Hence

\[(18) \quad EI(A) = \frac{1}{2} \lambda \sigma V alf E(l) E(f).\]

**Final remarks**

1. Note that for Equations (16), (17) and (18) the hypothesis of orientation that we attached to the segment process is irrelevant, and that it was adopted for ease of reference only.

2. The same general formulas of integral geometry can be used to study line segment processes in the plane. The treatment there is slightly simpler, for one gets directly the kinematic density in $R^2$.

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**References**


