J. Appl. Prob. **9**, 140–157 (1972)
Printed in Israel

AVERAGES FOR POLYGONS FORMED BY RANDOM LINES IN EUCLIDEAN AND HYPERBOLIC PLANES

L. A. SANTALÓ, *University of Buenos Aires*
I. YAÑEZ, *University of Madrid*

Abstract

We consider a countable number of independent random uniform lines in the hyperbolic plane (in the sense of the theory of geometrical probability) which divide the plane into an infinite number of convex polygonal regions. The main purpose of the paper is to compute the mean number of sides, the mean perimeter, the mean area and the second order moments of these quantities of such polygonal regions. For the Euclidean plane the problem has been considered by several authors, mainly Miles [4]–[9] who has taken it as the starting point of a series of papers which are the basis of the so-called stochastic geometry.

GEOMETRICAL PROBABILITY; RANDOM LINES; RANDOM POLYGONS; PLANE OF CONSTANT CURVATURE; HYPERBOLIC PLANE; CONVEX DOMAINS; POISSON LINE PROCESS

1. Introduction

Consider the Euclidean plane uniformly covered by random lines which will divide the plane into an infinite number of convex polygonal regions. This set of random polygonal regions was first studied by Goudsmit [2] who obtained the mean number of sides, the mean perimeter, the mean area and the mean area-squared of the polygons. More general results were obtained later by Miles ([4], [5]) and Richards [10]. Interesting generalizations to Euclidean n-dimensional space have been established by Miles ([6], [7], [8] and [9]).

In [12] one of the present authors studied the same problem for the hyperbolic plane. He considers first the regions into which a fixed circle of radius r is divided by n random lines and then takes the limit of the expected values corresponding to these regions as n and r tend to infinity in such a way that n/r tends to a finite constant.

This procedure is satisfactory for the Euclidean plane. However, for the hyperbolic plane a more detailed study is necessary. Consider, for instance, the plane divided into an infinite set of convex polygons by a countable number of independent random uniform lines and a circle of radius r placed on it (Figure 1). We may consider the mean area $E_r^*(A)$ of the *regions* into which the circle is

Received in revised form 19 March 1971.

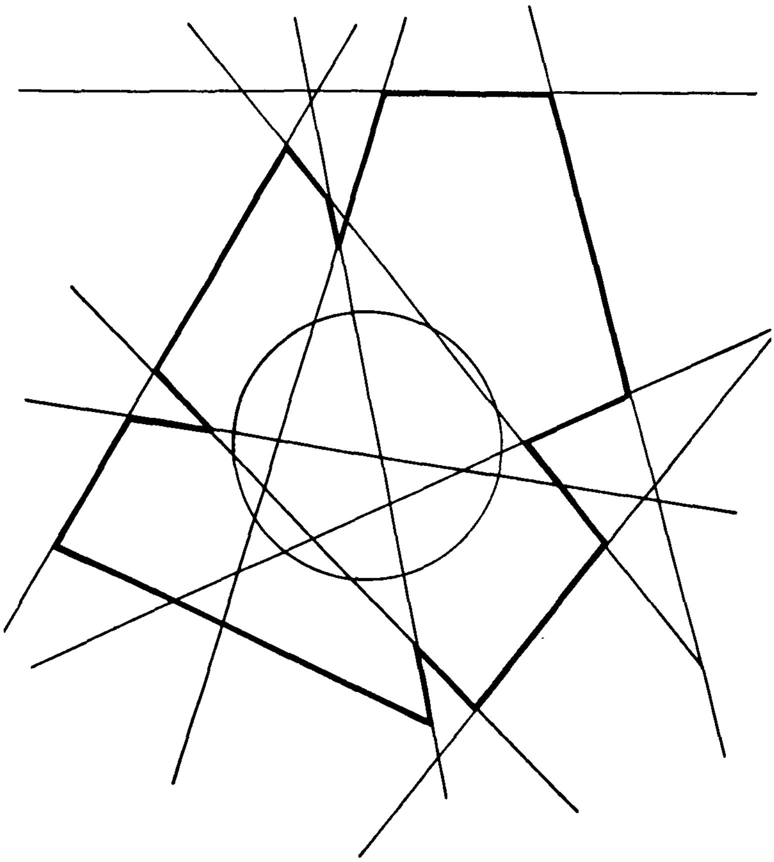

Figure 1

partitioned and the mean area $E_r(A)$ of the *polygons* having at least one point in common with the circle. In the case of Figure 1 we have the "empiric averages" $E_r^*(A) = \pi r^2/13$ and $E_r(A) = F/13$, where F is the total area of the polygons having at least one point in common with the circle. In the Euclidean plane $E_r^*(A)$ and $E_r(A)$ tend to the same limit as $r \to \infty$, while in the hyperbolic plane both limits have different values (which we will denote by $E^*(A)$ and $E(A)$ respectively). This distinction was missing in [12], where only the mean values E^* were considered. The difference arises from the fact that in the Euclidean plane the edge effects on the boundary of the circle may be disregarded and in the hyperbolic plane they may not.

In this paper we take the problem from the beginning and consider, from a general point of view, the plane of constant curvature $k \leq 0$, which will be denoted by $H(k)$. For $k = 0$ we have the Euclidean plane and for $k = -1$ we have the hyperbolic plane.

The lemmas of Section 3 show that, instead of the circle of the example above, we may take any convex domain which expands to the whole plane; the limits of the expected values do not depend on the shape of these convex domains.

2. Random lines in the Euclidean and hyperbolic plane: compilation of known formulas

The formulas and results of this section can be seen in [11].

Consider the plane of constant curvature $k \leq 0$. Let p, ϕ be the polar coordinates (or "geodesic" polar coordinates) of the foot of the perpendicular from the origin to the line G (Figure 2). Then, the density for the lines is

$$dG = \cos k^{\frac{1}{2}} p \, dp \wedge d\phi. \tag{2.1}$$

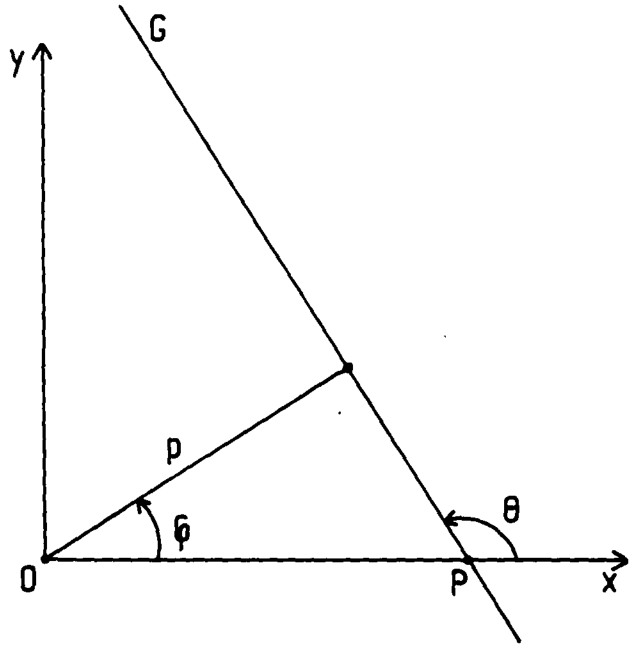

Figure 2

The measure of a set of lines is defined as the integral of dG over the set and it is the unique, up to a constant factor, which is invariant under motions in $H(k)$.

For the Euclidean plane, $k = 0$, we have

$$dG = dp \wedge d\phi \tag{2.2}$$

and for the hyperbolic plane, $k = -1$, since $\cos ix = \cosh x$, we have

$$dG = \cosh p \, dp \wedge d\phi \qquad (p \geq 0, \quad 0 \leq \phi \leq 2\pi). \tag{2.3}$$

If the line G is determined by the abscissa x of the intersection point P of G with a fixed line Ox through the origin O (Figure 2) and the angle θ between G and Ox, an easy change of coordinates yields

$$dG = \sin\theta \, dx \wedge d\theta \qquad (0 \leqq \theta \leqq \pi). \tag{2.4}$$

This expression for the density of lines holds for any k. However, for $k = -1$, there are lines G (forming a set of measure ∞) which do not intersect Ox, so that the coordinate system x, θ is not admissible for all the lines of the hyperbolic plane.

From (2.4) we deduce the following mean values referring to the angle θ between a fixed line and a random line G which intersects the line:

$$E(\theta) = \tfrac{1}{2}\int_0^\pi \theta \sin\theta \, d\theta = \tfrac{1}{2}\pi, \quad E(1/\sin\theta) = \tfrac{1}{2}\int_0^\pi d\theta = \tfrac{1}{2}\pi. \tag{2.5}$$

Notice, moreover that $\sigma^2(\theta) = \frac{1}{4}\pi^2 - 2$. These mean values will be used later.

With the density (2.1) one can prove that the measure of the set of lines which intersect a convex domain K_0 is equal to the perimeter L_0 of K_0 (= length of the boundary ∂K_0), that is

$$\int_{G \cap K_0 \neq \varnothing} dG = L_0. \tag{2.6}$$

Hence, if $K \subset K_0$ is a convex domain of perimeter L, the probability that a random line intersecting K_0 also intersects K is L/L_0 (independent of the position of K within K_0) and, therefore, given n random lines intersecting K_0, the number m of these lines hitting K has a binomial distribution $(n, L/L_0)$ and its mean value is

$$E_{K,K_0}(m) = \frac{nL}{L_0}. \tag{2.7}$$

Moreover, if s denotes the length of the chord determined by G on K_0, we have ([7], page 25)

$$\int_{G \cap K_0 \neq \varnothing} s \, dG = \pi F_0, \tag{2.8}$$

where F_0 is the area of K_0. Hence, the mean value of s is

$$E_{K_0}(s) = \pi F_0 / L_0. \tag{2.9}$$

Assuming that K_0 expands to the whole plane $H(k)$, in a sense that will be made precise in the next section, and that $n \to \infty$ in such a way that

$$\frac{n}{L_0} \to \tfrac{1}{2}\lambda, \quad \lambda = \text{constant} \tag{2.10}$$

the number of lines intersecting K is Poisson distributed with parameter $\frac{1}{2}\lambda L$, i.e., the probability that K is intersected by exactly m lines is (independently of the position of K in the plane)

$$P_m = (m!)^{-1}(\tfrac{1}{2}\lambda L)^m e^{-\lambda L/2}, \tag{2.11}$$

and the mean value of m is

$$E(m) = \tfrac{1}{2}\lambda L. \tag{2.12}$$

One says that a Poisson line system is generated in $H(k)$ or, following Miles [6], and taking (2.3) into account, that we have in $H(k)$ an isotropic homogeneous Poisson line process corresponding to a Poisson point process of intensity $\frac{1}{2}\lambda \cosh p$ in the (p, ϕ)-strip $(0 \leqq p \leqq \infty, 0 \leqq \phi \leqq 2\pi)$.

If K reduces to a line segment of unit length we have $L = 2$; hence, λ is equal to the mean number of lines which are intersected by an arbitrary segment of unit length. As a consequence we have that the points of intersection of an arbitrary line with the lines of the system of random lines constitute a Poisson process of parameter λ.

Other formulas that we shall use referring to planes of constant curvature are the following. The arc element ds on $H(k)$ has the form

$$ds^2 = d\rho^2 + k^{-1}\sin^2 k^{\frac{1}{2}}\rho\, d\phi^2, \tag{2.13}$$

where ρ, ϕ are geodesic polar coordinates. The element of area at the point $P(\rho, \phi)$ writes

$$dP = \frac{\sin k^{\frac{1}{2}}\rho}{k^{\frac{1}{2}}} d\rho \wedge d\phi. \tag{2.14}$$

From (2.13) and (2.14) it follows that the perimeter and the area of a circle of radius ρ are, respectively

$$L_C = \frac{2\pi}{k^{\frac{1}{2}}}\sin k^{\frac{1}{2}}\rho, \quad F_C = \frac{2\pi}{k}(1 - \cos k^{\frac{1}{2}}\rho). \tag{2.15}$$

Between the perimeter L and the area F of a convex domain K we have the isoperimetric inequality

$$L^2 - 4\pi F + kF^2 \geqq 0, \tag{2.16}$$

where the equality sign holds if and only if K is a circle.

Finally let us recall the following formula of Gauss-Bonnet for planes of constant curvature. Let K be a domain of $H(k)$ bounded by a single curve ∂K. If ∂K is smooth and κ_g denotes its geodesic curvature, the classical Gauss-Bonnet formula gives

$$\int_{\partial K} \kappa_g ds = 2\pi - kF. \tag{2.17}$$

If K is a convex polygon with N vertices and θ_h $(h = 1, 2, \cdots, N)$ are its interior angles, then the formula of Gauss-Bonnet takes the form

$$-kF = (N-2)\pi - \sum_{h=1}^{N} \theta_h \tag{2.18}$$

For (2.17) and (2.18) see, for instance, Guggenheimer ([3], page 283).

3. Two lemmas

Consider first the Euclidean plane $k = 0$. Let $K_0(t)$ be a family of convex domains depending upon the parameter t. Let $F_0(t)$ be the area and $L_0(t)$ the perimeter of $K_0(t)$. Assume that for any point P of the plane, there is a value t_P of t such that, for all $t > t_P$, we have $P \in K_0(t)$. That means that $K_0(t)$ expands over the whole plane $H(k)$ as $t \to \infty$.

Lemma 1. In the Euclidean plane we have

$$\lim_{t\to\infty} \frac{F_0(t)}{L_0(t)} = \infty \tag{3.1}$$

independently of the shape of $K_0(t)$.

Proof. Let $C(t)$ be the greatest circle contained in $K_0(t)$ and let $R(t)$ be its radius. Let O be the center of $C(t)$ and let $h_t = h_t(\phi)$ be the support function of $K_0(t)$ with respect to the origin O. We have

$$F_0(t) = \frac{1}{2} \int_{\partial K_0} h_t(\phi)\, ds_t, \tag{3.2}$$

where ds_t is the arc element of ∂K_0 at the contact point of the support line perpendicular to the direction ϕ. Since $h_t(\phi) \geq R(t)$ we get $F_0(t) \geq \frac{1}{2}R(t)L_0(t)$ and because $R(t) \to \infty$ as $t \to \infty$ we get (3.1). Notice that the limit of the ratio F_0/L_0^2 may have any value $\leq 1/4\pi$ depending on the shape of $K_0(t)$. Notice, also, that (3.1) is not necessarily true for non-convex domains.

Let us now consider the hyperbolic plane $H(-1)$. With the same conditions as above, the isoperimetric inequality (2.16) gives

$$\lim_{t\to\infty} \frac{F_0(t)}{L_0(t)} \leq 1. \tag{3.3}$$

For simplicity, in the case of the hyperbolic plane instead of convex domains, we shall always restrict ourselves to the so-called h-convex domains, or domains which are convex with respect to horocycles (i.e., such that for each pair of points A, B belonging to the domain, the entire segments of the two horocycles AB also belong to the domain) (see [13]). Any h-convex domain is convex, but the converse is not true. If the boundary ∂K_0 is smooth, the necessary and sufficient condition for h-convexity is that the curvature of ∂K_0 (geodesic curvature) satisfies the condition $\kappa_g \geqq 1$. For instance, the circles are all h-convex. The Gauss-Bonnet formula (2.17) then gives $\lim_{t\to\infty}(F_0/L_0) \geqq 1$ and hence, taking (3.3) into account we have that for all h-convex domains which expand to the whole hyperbolic plane,

$$\lim_{t\to\infty} \frac{F_0(t)}{L_0(t)} = 1. \tag{3.4}$$

Further, for h-convex domains the diameter D_0 and the perimeter L_0 satisfy the inequality $L_0 \geqq 4 \sinh\frac{1}{2}D_0$ (see [14]) and thus

$$\lim_{t\to\infty} \frac{D_0(t)}{L_0(t)} = 0. \tag{3.5}$$

Though we have proved (3.4) and (3.5) for h-convex domains with a smooth boundary, since any convex domain may be approximated by convex domains with smooth boundaries and F_0, L_0, D_0 are continuous functionals, it follows that (3.4) and (3.5) hold for any h-convex domain. We conjecture that (3.4) and (3.5) hold for any family of convex domains. (not necessarily h-convex) which expand to the whole hyperbolic plane. However the proof seems to be rather involved, so we shall restrict attention to h-convex domains. As a matter of fact it would be sufficient to consider the family of circles of radius t, but we think that it is worthwhile to point out the independence of the shape of $K_0(t)$ for the limits of Sections 4 and 5.

We can state the following result.

Lemma 2. In the hyperbolic plane, given a family of h-convex domains $K_0(t)$ such that $K_0(t)$ expands to the whole plane as $t \to \infty$, then the Relations (3.4) and (3.5) hold.

Notice that (3.1) and (3.4) can be condensed into

$$\lim_{t\to\infty} \frac{F_0(t)}{L_0(t)} = |k|^{-\frac{1}{2}}. \tag{3.6}$$

Using (2.15), one can easily verify (3.6) when $K_0(t)$ is a circle of radius t.

4. First mean values

Let K_0 be a convex domain of $H(k)$. Let F_0 be the area and L_0 the perimeter of K_0. Let $G_i\,(i = 1, 2, \cdots, n)$ be n lines which intersect K_0 and let V be the number of intersection points $G_i \cap G_j$ which are inside K_0. We want to compute the integral

$$I = \int_{G_h \cap K_0 \neq \emptyset} V\, dG_1 \wedge dG_2 \wedge \cdots \wedge dG_n \tag{4.1}$$

extended over all the lines G_h which intersect K_0 ($h = 1, 2, \cdots, n$). Let V_{ij} be the function of G_i, G_j which is equal to 1 if $G_i \cap G_j \in K_0$ and is equal to 0 otherwise (set $V_{ii} = 0$ for completeness). We have $V = \sum_{i<j} V_{ij}$ and

$$
\begin{aligned}
I &= \tfrac{1}{2}n(n-1) \int V_{ij}\, dG_1 \wedge dG_2 \wedge \cdots \wedge dG_n \\
&= \tfrac{1}{2}n(n-1)L_0^{n-2} \int V_{ij}\, dG_i \wedge dG_j = \tfrac{1}{2}n(n-1)L_0^{n-2}\, 2 \int s_i\, dG_i \qquad (4.2)\\
&= n(n-1)\pi F_0 L_0^{n-2},
\end{aligned}
$$

where s_i denotes the length of the chord $G_i \cap K_0$ and the integrals are extended over all the lines which intersect K_0.

Since, by (2.6)

$$
\int_{G_h \cap K \neq \varnothing} dG_1 \wedge dG_2 \wedge \cdots \wedge dG_n = L_0^n, \qquad (4.3)
$$

we obtain the following result:

Given independently and at random n lines which intersect a convex domain K_0, the expected number of intersection points of these lines which are interior to K_0, is

$$
E_{K_0,n}(V) = \pi n(n-1) \frac{F_J}{L_0^2}. \qquad (4.4)
$$

Note. Let $K_0(t)$ be a family of convex (or h-convex) domains which expand to the whole $H(k)$ as $t \to \infty$. For the Euclidean plane the limit of $E_{K_0(t),n}(V)$ as $t \to \infty$ depends on the shape of $K_0(t)$ and we can only affirm that it is at most $\frac{1}{4}n(n-1)$. On the other hand, for the hyperbolic plane, according to (3.4), the limit is 0, independently of the shape of $K_0(t)$ (assumed h-convex).

The same method leads to the evaluation of

$$
I_2 = \int_{G_h \cap K_0 \neq \varnothing} V^2 dG_1 \wedge dG_2 \wedge \cdots \wedge dG_n. \qquad (4.5)
$$

In fact we have $V^2 = (\sum_{i<j} V_{ij})^2 = \sum_{i<j} V_{ij}^2 + 2 \sum V_{ij} V_{kl}$, where in the second sum the range of indices assumes $i < j$, $k < l$ and the cases $i = k, j = l$ are excluded. Thus

$$
\begin{aligned}
I_2 &= \int V\, dG_1 \wedge \cdots \wedge dG_n + 2 \int \sum V_{ij} V_{kl}\, dG_1 \wedge \cdots \wedge dG_n \\
&= n(n-1)\pi F_0 L_0^{n-2} + 2L_0^{n-4} \binom{n}{4}\, 3 \int V_{ij} V_{kl}\, dG_i \wedge dG_j \wedge dG_k \wedge dG_l \qquad (4.6)\\
&\quad + 2L_0^{n-3} \binom{n}{3}\, 3 \int V_{ij} V_{il}\, dG_i \wedge dG_j \wedge dG_l,
\end{aligned}
$$

where in the first integral of the last term the factor 3 arises from the possibilities $V_{ij}V_{kl}$, $V_{ik}V_{jl}$, $V_{il}V_{jk}$ (assuming $i<j<k<l$) and in the second integral the factor 3 arises from the possibilities $V_{ij}V_{il}$, $V_{ij}V_{jl}$, $V_{il}V_{jl}$. Performing the integration we get

$$(4.7)\quad I_2 = n(n-1)\pi F_0 L_0^{n-2} + 24\binom{n}{4}\pi^2 F_0^2\, L_0^{n-4} + 24\binom{n}{3}L_0^{n-3}\int_{G\cap K_0\neq\varnothing} s^2 dG,$$

where s is the length of the chord $G\cap K_0$. Division by (4.3) yields

$$(4.8)\quad E_{K_0,n}(V^2) = \pi n(n-1)\frac{F_0}{L_0^2} + 24\pi^2\binom{n}{4}\frac{F_0^2}{L_0^4} + 24\binom{n}{3}L_0^{-3}\int_{G\cap K_0\neq\varnothing} s^2 dG.$$

If D_0 is the diameter of K_0 we have $\int s^2 dG \leqq D_0\int s\,dG = \pi F_0 D_0$ and therefore

$$(4.9)\qquad E_{K_0,n}(V^2) \leqq \pi n(n-1)\frac{F_0}{L_0^2} + 24\pi^2\binom{n}{4}\frac{F_0^2}{L_0^4} + 24\binom{n}{3}\pi\frac{F_0 D_0}{L_0^3}.$$

These formulas are valid for any convex domain of $H(k)$. Assume now, that in the case $k<0$, K_0 is h-convex. Furthermore suppose that K_0 is dependent on a parameter t and that $K_0(t)$ expands to the whole plane as $t\to\infty$. Assume further that n increases with t in such a way that

$$(4.10)\qquad \lim_{t\to\infty}\frac{n(t)}{L_0(t)} = \tfrac{1}{2}\lambda,$$

where λ is a constant. According to Section 2 we get an isotropic homogeneous Poisson line process of parameter λ, i.e., such that λ is equal to the mean number of lines intersecting an arbitrary segment of unit length.

For the random variable V/F_0, depending on t (and therefore on n, by (4.10)) we have, using (4.9) and (3.5),

$$(4.11)\qquad \lim_{t\to\infty} E_{K_0,n}\left(\frac{V}{F_0}\right) = \lim_{t\to\infty}\frac{\pi n(n-1)}{L_0^2} = \tfrac{1}{4}\pi\lambda^2$$

and

$$(4.12)\qquad \lim_{t\to\infty}\sigma^2_{K_0,n}\left(\frac{V}{F_0}\right) = \lim_{t\to\infty}\left[E_{K_0,n}\left(\frac{V^2}{F_0^2}\right) - E^2_{K_0,n}\left(\frac{V}{F_0}\right)\right] = 0.$$

It follows that, for any convex (or h-convex) domain $K_0(t)$ which expands to the whole plane we have, with probability one,

$$(4.13)\qquad \lim_{t\to\infty}\frac{V}{F_0} = \tfrac{1}{4}\pi\lambda^2.$$

From (4.13), (4.10) and (3.6) we get, with probability one,

$$(4.14)\qquad \lim_{t\to\infty}\frac{n}{V} = \lim_{t\to\infty}\frac{n}{L_0}\,\frac{L_0}{F_0}\,\frac{F_0}{V} = \frac{2|k|^{\frac{1}{2}}}{\pi\lambda}.$$

Note, for later use, that (4.13) gives the mean number of intersection points of two lines per unit area in $H(k)$ and therefore the probability that a random element of area $d\sigma$ in the plane contains an intersection point of two lines, is

$$\tfrac{1}{4}\pi\lambda^2 d\sigma. \tag{4.15}$$

5. Mean values for the regions into which a convex (or h-convex) domain is divided by random lines

Consider the plane of constant curvature k and a convex (or h-convex) domain $K_0 = K_0(t)$ in it. We desire to study some mean values concerning the regions into which K_0 is divided by n random lines G_i which intersect K_0. Assume that there are not three or more lines intersecting in a common point of K_0 (according to the measure defined in Section 2, these lines form a set of measure zero). The chords $G_i \cap K_0$ and the boundary ∂K_0 form a plane graph which has $V + 2n$ vertices (V vertices which are interior to K_0 and $2n$ vertices on ∂K_0). The number of sides of the graph is clearly $\frac{1}{2}(4V + 6n) = 2V + 3n$. Therefore, calling P the number of regions into which K_0 is partitioned by the random lines, the classical Euler's formula (regions $-$ sides $+$ vertices $= 1$) gives $P - 2V - 3n + V + 2n = 1$ and we get

$$P = V + n + 1. \tag{5.1}$$

For instance, in the case of Figure 1 we have $n = 6$, $V = 6$, $P = 13$, number of sides $= 30$.

The equality (5.1) and (4.4) give

$$E_{K_0, n}(P) = \pi n(n-1) \frac{F_0}{L_0^2} + n + 1, \tag{5.2}$$

which is *the mean value of the number of regions into which a convex domain K_0 is partitioned by n random lines which cross it.*

Assuming that $K_0(t)$ expands to the whole plane $H(k)$ as $t \to \infty$ with the Condition (4.10), the mean number of regions per unit area will be (using (4.13) and (3.6))

$$\lim_{t\to\infty} \frac{P}{F_0} = \lim_{t\to\infty} \left(\frac{V}{F_0} + \frac{n}{L_0}\frac{L_0}{F_0} + \frac{1}{F_0} \right) = \tfrac{1}{4}\pi\lambda^2 + \tfrac{1}{2}\lambda |k|^{\frac{1}{2}} \tag{5.3}$$

and hence the limit of the mean area A of the regions into which K_0 is partitioned is

$$E^*(A) = \frac{4}{\pi\lambda^2 + 2\lambda |k|^{\frac{1}{2}}}. \tag{5.4}$$

As in (4.11) and (4.14) the limit (5.3) (and hence (5.4)), like the following limits (5.6) and (5.7) are limits "in probability", i.e., the equality occurs with probability one.

Let now N_i be the number of vertices of the ith region ($i = 1, 2, \cdots, P$). We have

$$\sum_{i=1}^{P} N_i = 4V + 4n \tag{5.5}$$

and therefore the limit of the mean number of vertices of a region with the boundary regions included, is

$$E^*(N) = \lim_{t\to\infty} \frac{4V + 4n}{P} = \lim_{t\to\infty} \frac{4V + 4n}{V + n + 1} = 4. \tag{5.6}$$

Finally, if s_i is the length of the chord $G_i \cap K_0$, the sum of the perimeters of the regions, for a given set of lines $G_1, G_2, \cdots, G_n$ which cut K_0 is $2\sum s_i + L_0$ ($i = 1, 2, \cdots, n$) and thus

$$E^*(L) = \lim_{t\to\infty} \frac{2\sum s_i + L_0}{P} = \lim_{t\to\infty} \frac{2\sum s_i/n + L_0/n}{V/n + 1 + 1/n} \tag{5.7}$$

and, since $\lim_{t\to\infty}(L_0/n) = 2/\lambda$, $\lim_{t\to\infty}(V/n) = \pi\lambda/2|k|^{\frac{1}{2}}$, $\lim_{t\to\infty}[\sum s_i/n]$ = limit of the mean length of the chords $G_i \cap K_0 = \lim_{t\to\infty}(\pi F_0/L_0) = \pi/|k|^{\frac{1}{2}}$, we get

$$E^*(L) = \frac{4\pi\lambda + 4|k|^{\frac{1}{2}}}{\pi\lambda^2 + 2\lambda|k|^{\frac{1}{2}}}. \tag{5.8}$$

The limit second order moments $E^*(A^2)$, $E^*(AL)$, $E^*(L^2), \cdots$ for regions, are known for the Euclidean plane, because they coincide with the second order moments for polygons and they have been given by Miles ([4], [5]). For the hyperbolic plane we only know $E^*(A^2)$ and we have $E^*(A^2) < \infty$ if and only if $\lambda > \frac{1}{2}$(cf. [12]). Moreover, for the hyperbolic plane we do not know if these limit second order moments are dependent or not on the shape of the expanding domain $K_0(t)$.

6. Mean values for polygons determined by random lines

Consider the Poisson line system described in Sections 2 and 4, which partitions $H(k)$ into an aggregate T of random convex polygons. Our object is now to investigate some mean values of certain quantities Z attached to each polygon, the basic ones being the area A, the perimeter L and the number of sides (or vertices) N.

To this end, the natural way should be to consider first the subaggregate of polygons T_t having at least one point in common with $K_0(t)$ and then make $t \to \infty$. However in this case it seems to be difficult to compute directly the empiric averages $\sum_t Z/\sum_t 1$ ($\sum_t Z$ = sum of Z-values for the polygons of T_t) and we must follow an indirect method which makes necessary the introduction of certain assumptions.

Let $F_{Z,t}(z)$ (z is a particular value of Z) be the empiric distribution function of Z for the finite aggregate T_t ($F_{Z,t}(z)$ depends on the way of selecting the random polygons). Then we make the assumption that for each Z there exists a distribution function $F_Z(z)$ such that $F_{Z,t}(z)$ tends almost surely to $F_Z(z)$ as $t \to \infty$ and, moreover, the empiric averages $\Sigma_t Z / \Sigma_t 1$ converge almost surely to the ergodic mean $\int Z F_Z(dz)$.

For details about these assumptions and their proof for the Euclidean plane, together with some deep reasons for their plausibility in general, see Miles ([4], [5], [6] and [8]).

We will first consider the following mean values, depending of the way of selecting the random polygons.

(a) Select at random a point on the plane and consider the value of Z of the polygon which contains the point. The corresponding mean value will be denoted by $E_A(Z)$, $Z = A, L, N$.

(b) Select a vertex at random (i.e., an intersection point of two lines) and, with probability $\frac{1}{4}$, select one of the four polygons having this vertex. This procedure gives rise to the mean values which we will denote by $E_N(Z)$.

(c) Select at random a point on one of the lines and then, with probability $\frac{1}{2}$, select one of the two polygons which contain the point in its boundary. We call $E_L(Z)$ the corresponding mean value.

Of course these mean values presuppose that the stated selections are meaningful, in particular, the assumption that it is meaningful to select a random vertex of T.

We proceed to compute these mean values.

Method (a). Given a random line G, the intersection points of G with the lines $G_1, G_2, \cdots$ of the Poisson line system which determines T are distributed according to a Poisson distribution of parameter λ. Therefore, given a random point P on the plane and a random line G through P, the probability that the distance from P to the first intersection point of G with T lies in the interval ρ $\rho + d\rho$ ($\rho \geq 0$), is

$$(6.1) \qquad \lambda e^{-\lambda\rho} d\rho.$$

According to (2.14) the area of the polygon which contains P will be

$$(6.2) \qquad A = \int_0^{2\pi} \int_0^{\rho} \frac{\sin k^{\frac{1}{2}}\rho}{k^{\frac{1}{2}}} d\rho \wedge d\phi = k^{-1} \int_0^{2\pi} (1 - \cos k^{\frac{1}{2}}\rho) d\phi,$$

and therefore, by (6.1),

$$(6.3) \qquad E_A(A) = k^{-1} \int_0^{2\pi} \int_0^{\infty} \lambda e^{-\lambda\rho}(1 - \cos k^{\frac{1}{2}}\rho) d\rho \wedge d\phi.$$

Performing the integration we get

(6.4) $$E_A(A) = \infty \text{ for } \lambda \leq |k|^{\frac{1}{2}},\ E_A(A) = \frac{2\pi}{\lambda^2 + k} \quad \text{for } \lambda > |k|^{\frac{1}{2}}.$$

In order to compute $E_A(L)$ we apply (2.13) and calling α the angle of G with the side of the polygon at the intersection point (Figure 3), we have

(6.5) $$L = \int_0^{2\pi} \frac{\sin k^{\frac{1}{2}}\rho}{k^{\frac{1}{2}}} \frac{1}{\sin\alpha} d\phi.$$

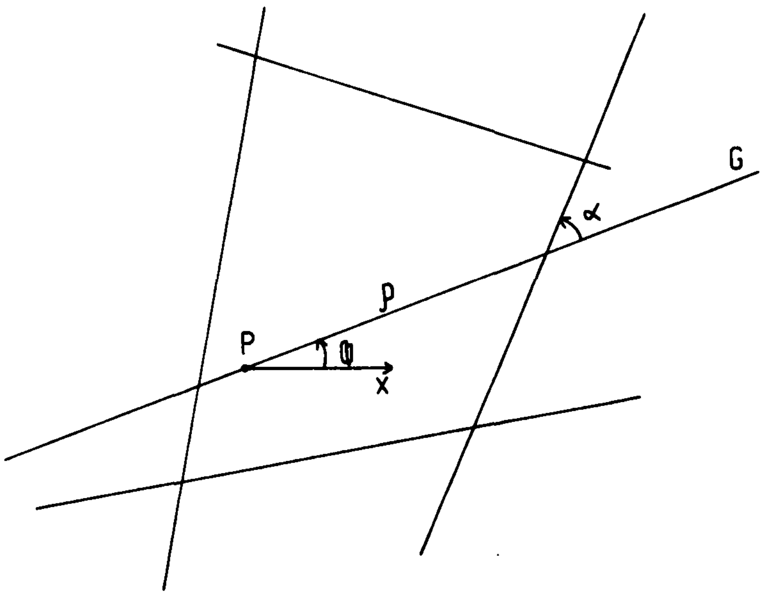

Figure 3

Since ρ and α are independent, taking (2.5) into account, we have

(6.6) $$\begin{aligned} E_A(L) &= \int_0^{2\pi} E\left(\frac{\sin k^{\frac{1}{2}}\rho}{k^{\frac{1}{2}}}\right) E\left(\frac{1}{\sin\alpha}\right) d\phi \\ &= (\pi\lambda/2k^{\frac{1}{2}}) \int_0^{2\pi}\int_0^{\infty} e^{-\lambda\rho} \sin k^{\frac{1}{2}}\rho \, d\rho \wedge d\phi, \end{aligned}$$

and therefore

(6.7) $$E_A(L) = \infty \text{ for } \lambda \leq |k|^{\frac{1}{2}},\ E_A(L) = \frac{\pi^2\lambda}{\lambda^2 + k} \quad \text{for } \lambda > |k|^{\frac{1}{2}}.$$

We now proceed to compute $E_A(N)$. Consider the random variable $u(d\sigma(Q))$ associated to each area element $d\sigma(Q)$ of the plane, defined as 1 if the area element $d\sigma(Q)$ contains a vertex of T and the segment PQ does not intersect any line of T, and 0 otherwise. According to (4.15) and since the probability that a segment

PQ of length ρ does not contain any point in a Poisson process with parameter λ is equal to $e^{-\lambda\rho}$, we have

$$E_A(N) = \tfrac{1}{2} \int_0^{2\pi} \int_0^{\infty} \pi\lambda^2 e^{-\lambda\rho} d\sigma = \tfrac{1}{2}k^{-\frac{1}{2}}\pi^2\lambda^2 \int_0^{\infty} e^{-\lambda\rho} \sin k^{\frac{1}{2}}\rho\, d\rho \tag{6.8}$$

or

$$E_A(N) = \infty \text{ for } \lambda \leqq |k|^{\frac{1}{2}},\ E_A(N) = \frac{\pi^2\lambda^2}{2(\lambda^2 + k)} \quad \text{for } \lambda > |k|^{\frac{1}{2}}. \tag{6.9}$$

Method (b). We now select at random a vertex Q of the system of random polygons T. By (2.14) if θ denotes the interior angle of the polygon corresponding to the vertex Q, we have

$$A = k^{-1} \int_0^{\theta} (1 - \cos k^{\frac{1}{2}}\rho)\, d\phi \tag{6.10}$$

and thus, by (6.1) and (2.4) we have

$$\begin{aligned} E_N(A) &= k^{-1} \int_0^{\pi} \tfrac{1}{2} \left[\int_0^{\theta} \int_0^{\infty} \lambda e^{-\lambda\rho} (1 - \cos k^{\frac{1}{2}}\rho) d\rho \wedge d\phi \right] \sin\theta\, d\theta \\ &= (2k)^{-1} \int_0^{\infty} \lambda e^{-\lambda\rho} (1 - \cos k^{\frac{1}{2}}\rho)\, d\rho \int_0^{\pi} \theta \sin\theta\, d\theta \\ &= \tfrac{1}{2} E_A(A) \end{aligned} \tag{6.11}$$

and thus

$$E_N(A) = \infty \text{ for } \lambda \leqq |k|^{\frac{1}{2}},\ E_N(A) = \frac{\pi}{2(\lambda^2 + k)} \quad \text{for } \lambda > |k|^{\frac{1}{2}}. \tag{6.12}$$

To compute $E_N(L)$ we observe that the sides opposite to the chosen vertex Q, by similar considerations as above, give the term $\frac{1}{2}E_A(L)$ and the sides which are adjacent to Q give the mean length $1/\lambda$. Thus we have, $E_N(L) = \infty$ for $\lambda \leqq |k|^{\frac{1}{2}}$ and

$$E_N(L) = \tfrac{1}{2}E_A(L) + 2/\lambda = \frac{(\pi^2 + 8)\lambda^2 + 8k}{4\lambda(\lambda^2 + k)} \quad \text{for } \lambda > |k|^{\frac{1}{2}}. \tag{6.13}$$

Similarly, we get $E_N(N) = \infty$ for $\lambda \leqq |k|^{\frac{1}{2}}$ and

$$E_N(N) = \tfrac{1}{2}E_A(N) + 3 = \frac{(\pi^2 + 24)\lambda^2 + 24k}{8(\lambda^2 + k)} \quad \text{for } \lambda > |k|^{\frac{1}{2}}, \tag{6.14}$$

where the term $E_A(N)$ stands for the vertices which are opposite to Q and the term 3 stands for the two vertices adjacent to Q and for Q itself.

Method (c). By similar considerations as above we get the remaining mean values:

(6.15) $$E_L(A) = \tfrac{1}{2}E_A(A) = \frac{\pi}{\lambda^2 + k}, \quad \lambda > |k|^{\frac{1}{2}}$$

(6.16) $$E_L(L) = \tfrac{1}{2}E_A(L) + 2/\lambda = \frac{(\pi^2 + 4)\lambda^2 + 4k}{2\lambda(\lambda^2 + k)}, \quad \lambda > |k|^{\frac{1}{2}}$$

(6.17) $$E_L(N) = \tfrac{1}{2}E_A(N) + 2 = \frac{(\pi^2 + 8)\lambda^2 + 8k}{4(\lambda^2 + k)}, \quad \lambda > |k|^{\frac{1}{2}}.$$

For $\lambda \leqq |k|^{\frac{1}{2}}$ these mean values are all ∞.

Notice that in the case of the hyperbolic plane, $k = -1$, in order to obtain a finite value of the means E_A, E_N, E_L we must have $\lambda > 1$.

Alternative proof. We are indebted to the referee for the remark that all the preceding mean values can be deduced from $E_A(A)$ using a very ingenious device due to Miles [5]. Indeed, select a random vertex of T, say Q, and denote by $P_1, \cdots, P_4$, in order about Q, the four polygons having Q as common vertex. Then $P_{1234} = P_1 \cup P_2 \cup P_3 \cup P_4$ is a random polygon selected by method (a); $P_{12} = P_1 \cup P_2$, $P_{34} = P_3 \cup P_4$ are random polygons selected by method (c) and P_1, P_2, P_3, P_4 are random polygons selected by method (b). Using these remarks one can deduce some relations between the means E_A, E_N, E_L which determine all of them from $E_A(A)$. Though this method is more elegant and shorter, the proofs given above are perhaps more natural and straightforward.

7. The second moments

Let $dF_A(A,N,L)$ denote the probability that the polygon chosen according to method (a) has area between A and $A + dA$, perimeter between L and $L + dL$ and has N vertices. Similarly, let $dF_N(A,N,L)$ and $dF_L(A,N,L)$ denote the analogous probabilities for the polygons chosen with methods (b) and (c). We have made the assumption that all such probabilities exist.

Let $dF(A,N,L)$ denote the probability that a random polygon of T has area between A and $A + dA$, perimeter between L and $L + dL$ and has N vertices. Here the method of sampling a random polygon has the following sense: we consider first the finite number of polygons which intersect a convex (or h-convex) domain $K_0(t)$, all equally likely; for this finite set the probability $dF_t(A,N,L)$ has a well-defined meaning and then $dF(A,N,L)$ is the limit value of this probability as $t \to \infty$, so that $K_0(t)$ expands to the whole plane $H(k)$. Our assumption is that this limit exists almost surely.

Since the probability that a randomly chosen point on the plane belongs to a polygon of area A is proportional to A (i.e., the probability of choosing a polygon of area A by method (a) is proportional to A), we have the equation

(7.1) $$dF_A(A,N,L) = \frac{A\,dF(A,N,L)}{E(A)},$$

and similarly

$$(7.2)\qquad dF_N(A,N,L) = \frac{N\,dF(A,N,L)}{E(N)},\quad dF_L(A,N,L) = \frac{L\,dF(A,N,L)}{E(L)}.$$

Multiplying (7.1) and (7.2) by A, L, N and integrating over all values of A, N, L we get

$$(7.3)\qquad E(A^2) = E(A)\,E_A(A),\quad E(LA) = E(A)\,E_A(L),\quad E(NA) = E(A)\,E_A(N),$$

$$(7.4)\qquad E(AN) = E(N)\,E_N(A),\quad E(LN) = E(N)\,E_N(L),\quad E(N^2) = E(N)\,E_N(N),$$

$$(7.5)\qquad E(AL) = E(L)\,E_L(A),\quad E(L^2) = E(L)\,E_L(L), E(NL) = E(L)\,E_L(N).$$

From these relations we deduce

$$(7.6)\qquad \begin{aligned} E(A)\,E_A(L) &= E(L)\,E_L(A),\\ E(N)\,E_N(A) &= E(A)\,E_A(N),\\ E(L)\,E_L(N) &= E(N)\,E_N(L), \end{aligned}$$

which yield the identity

$$(7.7)\qquad E_A(L)\,E_N(A)\,E_L(N) = E_L(A)\,E_A(N)\,E_N(L)$$

which can be directly verified from the results of Section 6.

The identity (7.7) shows that the Equations (7.6) are not independent and thus they are not sufficient for computing the mean values $E(A)$, $E(N)$, $E(L)$. It seems very plausible that a new relation must be given by the Gauss-Bonnet formula (2.18). Indeed calling A_i, N_i the area and the number of vertices of the polygons having at least one point in common with $K_0(t)$ $(i = 1,2,\cdots,P(t))$, using (2.18) we have

$$(7.8)\qquad -kA_i = (N_i - 2)\pi - \sum_{h=1}^{N_i} \theta_{ih},\qquad (i = 1,2,\cdots,P(t))$$

where θ_{ih} $(h = 1,2,\cdots,N_i)$ are the interior angles of the ith polygon. Since the total angle at a point is 2π we may accept that, almost surely,

$$(7.9)\qquad \sum_{i,h} \theta_{ih} \text{ is equivalent to } \tfrac{1}{2}\pi \sum_i N_i \text{ as } t \to \infty.$$

This assumption supposes that "edge effects" are negligible, a fact that is surely true for the Euclidean plane, but which should be interesting to prove for the hyperbolic plane. Accepting (7.9), then (7.8) gives

$$(7.10)\qquad \pi E(N) + 2kE(A) = 4\pi.$$

This equation, together with the system (7.6) and the values of Section 6 give the result

(7.11) $E(A) = \dfrac{4\pi}{\pi^2\lambda^2 - 2|k|}$, $E(L) = \dfrac{4\pi^2\lambda}{\pi^2\lambda^2 - 2|k|}$, $E(N) = 4 + \dfrac{8|k|}{\pi^2\lambda^2 - 2|k|}$

for $\lambda > (2|k|)^{\frac{1}{2}}/\pi$, and $E(A)$, $E(N)$, $E(L) = \infty$ for $\lambda \leqq (2|k|)^{\frac{1}{2}}/\pi$. Since $k \leqq 0$ in the expressions above we can put $|k| = -k$.

Taking (7.11) into account, the Equations (7.3), (7.4) and (7.5) give the following second order moments. For $\lambda > 1$,

$$E(A^2) = \frac{8\pi^2}{(\pi^2\lambda^2 - 2|k|)(\lambda^2 + k)}, \tag{7.12}$$

$$E(AL) = \frac{4\pi^3\lambda}{(\pi^2\lambda^2 - 2|k|)(\lambda^2 + k)}, \tag{7.13}$$

$$E(AN) = \frac{2\pi^3\lambda^2}{(\pi^2\lambda^2 - 2|k|)(\lambda^2 + k)}, \tag{7.14}$$

$$E(L^2) = \frac{2\pi^2[(\pi^2 + 4)\lambda^2 + 4k]}{(\pi^2\lambda^2 - 2|k|)(\lambda^2 + k)}, \tag{7.15}$$

$$E(NL) = \frac{\pi^2\lambda[(\pi^2 + 8)\lambda^2 + 8k]}{(\pi^2\lambda^2 - 2|k|)(\lambda^2 + k)}, \tag{7.16}$$

$$E(N^2) = \frac{\pi^2\lambda^2[(\pi^2 + 24)\lambda^2 + 24k]}{2(\pi^2\lambda^2 - 2|k|)(\lambda^2 + k)}, \tag{7.17}$$

and for $\lambda \leqq 1$ the six second order moments become ∞.

Notes. 1. For the Euclidean plane $k = 0$, the values (7.11) coincide with the values (5.4), (5.6) and (5.8), that is, we have $E^* = E$. These values agree with those given by Miles [4]. Also the second order moments (7.12), $\cdots$, (7.17), for $k = 0$ agree with those given by Miles [4].

2. For the Euclidean plane the first and second order moments of A, N, L are finite for any $\lambda > 0$. For the hyperbolic plane the first order moments are finite if and only if $\lambda > 2^{\frac{1}{2}}/\pi = 0.450$, and the second order moments are finite if and only if $\lambda > 1$. We do not know if these critical values of λ increase with higher order moments.

The authors are indebted to the referee for many valuable suggestions and improvements.

References

[1] Feller, W. (1966) *An Introduction to Probability Theory and its Applications.* Vol. II. J. Wiley, New York.

[2] Goudsmit, S. A. (1945) Random distributions of lines in a plane. *Rev. Modern Phys.* 17, 321–322.

[3] Guggenheimer, H. (1963) *Differential Geometry.* McGraw Hill, New York.

[4] Miles, R. E. (1964) Random polygons determined by random lines in a plane. *Proc. Nat. Acad. Sci.* **52**, 901–907.

[5] Miles, R. E. (1964) Random polygons determined by random lines in a plane (II). *Proc. Nat. Acad. Sci.* **52**, 1157–1160.

[6] Miles, R. E. (1970) A synopsis of Poisson flats in Euclidean spaces. *Izv. Akad. Nauk Armjan. SSR Ser. Mat.* V, 3, 263–285.

[7] Miles, R. E. (1969) Poisson flats in Euclidean spaces, Part I: A finite number of random uniform flats. *Adv. Appl. Prob.* **1**, 211–237.

[8] Miles, R. E. (1971) Poisson flats in Euclidean spaces, Part II: Homogeneous Poisson flats and the complementary theorem. *Adv. Appl. Prob.* **3**, 1–43.

[9] Miles, R. E. (1970) On the homogeneous planar Poisson point process. *Math. Biosciences* **6**, 85–127.

[10] Richards, P. I. (1964) Averages for polygons formed by random lines. *Proc. Nat. Acad. Sci.* **52**, 1160–1164.

[11] Santaló, L. A. (1953) *Introduction to Integral Geometry*. Hermann, Paris.

[12] Santaló, L. A. (1966) Valores medios para polígonos formados por rectas al azar en el plano hiperbólico. *Rev. Mat. y Fis. Teor. Univ. Tucumán* **16**, 29–43.

[13] Santaló, L. A. (1967) Horocycles and convex sets in hyperbolic plane. *Arch. Math. (Basel)* **18**, 529–533.

[14] Santaló, L. A. (1969) Convexidad en el plano hiperbólico. *Rev. Mat. y Fis. Teor. Univ. Tucuman* **19**, 173–183.