hypocycloid. Prove that this hypocycloid is the locus of a point on a circle, radius $R/2 \sin \theta$, which rolls inside another circle, radius three times that of the rolling circle, whose center X is equidistant from the circumcenter C and the orthocenter O, and is such that angle $OXC = 2\theta$, R being the circumradius.

4116. *Proposed by N. A. Court, University of Oklahoma*

Given the tetrahedron $(T_1) = SA_1B_1C_1$, the tangent plane to its circumsphere at the diametric opposite of S meets the edges SA_1, SB_1, SC_1 in the points A_2, B_2, C_2. The tangent plane to the circumsphere of the tetrahedron $(T_2) = SA_2B_2C_2$ at the diametric opposite of S meets the edges of (T_2) through S in the points A_3, B_3, C_3 thus forming the tetrahedron $(T_3) = SA_3B_3C_3$, *etc.* Find the locus of the incenters of these tetrahedrons.

4117. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A polygon $A_1A_2 \cdots A_n$ may be transformed into a polygon $B_1B_2 \cdots B_n$ by locating the points B_i on the sides A_iA_{i+1} so that the ratio of A_iB_i to B_iA_{i+1} is equal to a constant r. Prove that if T_1 and T_2 are two such transformations for the ratios r_1 and r_2, then $T_1 \circ T_2 = T_2 \circ T_1$, and generalize.

4118. *Proposed by Otto Dunkel, Washington University*

Show that

$$\sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+4}}{t!(n-t)!} = \frac{(n+4)(n+3)\cdots n}{6!8}[15n^3 + 30n^2 + 15n - 2], \quad n \geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

4119. *Proposed by V. Thébault, San Sebastián, Spain*

The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the respective opposite sides meet in a point P. Show that the six points of contact of circles tangent to two sides and orthogonal to a given circle with center P are on a circle concentric with the inscribed circle.

SOLUTIONS

An Inequality for Triangles

4070 [1943, 124]. *Proposed by P. Erdös, Princeton, N. J.*

Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho + r \leqq m$.

Correction. The proposer intended to exclude obtuse angled triangles.

I. *Solution by L. A. Santaló, Rosario, Argentina.* Let $A_1A_2A_3$ be a triangle with no obtuse angle and with angles $A_1 \geqq A_2 \geqq A_3$; let O, I, H_i, B_i be the circumcenter, incenter, foot of the altitude from A_i, point of contact of incircle (I)

with side $a_i = A_jA_k$. If $I \equiv O$ the triangle is equilateral and $\rho + r = m$. If $A_1 = \pi/2$ and $A_1A_2 = A_1A_3$, then $\rho + r = m$.

Since $A_2B_1 \leqq B_1A_3$ and angle $OA_3B_1 \leqq$ angle IA_3B_1, the center O does not lie outside the triangle B_1A_3I. Also A_3O and A_3H_3 are symmetric with respect to A_3I. Let O_3 on A_3H_3 be the symmetric of O with respect to A_3I so that $A_3O_3 = r$, and let I_3 be the orthogonal projection of I on A_3H_3 so that $I_3H_3 = \rho$. Since angle $O_3IB_3 > \pi/2$, the point O_3 lies on segment A_3I_3, and it follows that $\rho + r \leqq A_3H_3 = m$.

If $A_1A_2A_3$ has an obtuse angle it is not easy to determine the relation between m and $\rho + r$. If $A_1A_2 = A_1A_3$ and angle $A_1 \to \pi$, then $m \to 0$; hence there are triangles for which $\rho + r > m$. On the contrary, if $A_1 > \pi/2$, $A_1A_2 < A_1A_3$ with A_1A_3 fixed, and we let $A_1 \to \pi/2$ and $A_1A_2 \to 0$; then we see that there are obtuse angled triangles for which $\rho + r < m$.

II. *Solution by Alfred Brauer and I. S. Cohen, University of North Carolina.* It is easy to see that this theorem is not always true. For example, let the triangle be isosceles and let its vertex angle approach 180°. Then $\rho \to 0$, $r \to \infty$, $m \to 0$, so that $\rho + r \leqq m$ cannot hold. The proposer subsequently indicated that the triangle should have been acute.

In the following, we prove, in fact, considerably more: Let the angles A, B, C of the triangle be such that

$$A \leqq B \leqq C. \tag{1}$$

Then the above theorem is true if $B \geqq 45°$; on the other hand it is definitely false if $B < 2 \text{ arc sin } \frac{1}{2}(\sqrt{3} - 1) = 42°\ 56' +$. If $42°\ 56' + < B < 45°$, then the theorem may be true or false, and it is definitely false if also $C \geqq 135°$. For an acute triangle, the theorem then follows from the fact that then $B \geqq 45°$.

Since A is the smallest angle, the longest altitude will be the one drawn from the vertex A. It can be shown that

$$\frac{m - \rho - r}{r} = \cos (C - B) - \cos C - \cos B = f(B, C).$$

It follows from (1) that

$$B \leqq C, \qquad B + C < 180°, \qquad 2B + C \geqq 180°.$$

These inequalities define in the (C, B)-plane a certain triangular domain, and we are interested in the values of $f(B, C)$ in this domain.

For a fixed B $(0 < B < 90°)$, let $f(B, C)$ be considered as a function of C in $B \leqq C \leqq 180°$. Then

$$\frac{\partial f}{\partial C} = - \sin (C - B) + \sin C$$

vanishes if and only if

$$C = 90° + \tfrac{1}{2}B,$$

and it is easily verified that this gives a maximum of f; moreover this is the only maximum in $B \leqq C \leqq 180°$ (for fixed B).

We now distinguish the cases $B \geqq 60°$ and $B < 60°$. If $B \geqq 60°$, then for the significant values of C we have

$$B \leqq C \leqq 180° - B \leqq 90° + \tfrac{1}{2}B.$$

Therefore, for fixed B, $f(B, C)$ is increasing in the significant interval, and so $f(B, C) \geqq f(B, B) = 1 - 2 \cos B \geqq 0$. Thus the theorem is proved when $B \geqq 60°$.

If, now, $45° \leqq B < 60°$, then the significant values of C are defined by $180° - 2B \leqq C < 180° - B$. Then the maximum $C = 90° + \frac{1}{2}B$ lies in this interior of the interval, and we must consider the function at both endpoints. Now at the left endpoint, $f(B, 180° - 2B) = \cos 2B(1 - 2\cos B) \geqq 0$. At the right endpoint, $f(B, 180° - B) = -\cos 2B \geqq 0$. The theorem is now proved for all triangles for which $B \geqq 45°$.

To see when the theorem will not be true, we note that it will certainly be false for those values of B for which $f(B, C)$ is negative at the maximum $C = 90° + \frac{1}{2}B$. At this maximum we have $f(B, 90° + \frac{1}{2}B) = 2 \sin^2 \frac{1}{2}B + 2 \sin \frac{1}{2}B - 1$. Since the roots of the quadratic $2x^2 + 2x - 1$ are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$, we have $f(B, 90° + \frac{1}{2}B) < 0$, if $-\frac{1}{2} - \frac{1}{2}\sqrt{3} < \sin \frac{1}{2}B < -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, that is, if

$$B < 2 \text{ arc sin } \tfrac{1}{2}(\sqrt{3} - 1) = 42°56' + .$$

Thus the theorem is certainly false if B is less than this angle.

If $42°\ 56' + \leqq B < 45°$, the theorem may be true or false, depending on the value of C. We show that if $C \geqq 135°$, then it is false. Namely,

$$f(B, C) = \cos (C - B) - 2 \cos \tfrac{1}{2}(C + B) \cos \tfrac{1}{2}(C - B).$$

Since $C \geqq 135°$, B is $< 45°$, it follows that $C - B > 90°$, and $\cos (C - B) < 0$. Since $\frac{1}{2}(C + B)$ and $\frac{1}{2}(C - B)$ are between 0° and 90°, the second term is also negative.

Solved also by H. Eves and I. Kaplansky.

Editorial Note. The solution by Kaplansky used the function $f(B, C)$ above with somewhat similar results. Eves showed that the theorem is not true for all obtuse triangles; and, for right and acute angled triangles, he gave a synthetic proof based on the equality $x_1 + x_2 + x_3 = \rho + r$ given in Johnson's *Modern Geometry*, art. 298, f. where the x_i's are absolute normal coordinates of the circumcenter of any triangle $A_1A_2A_3$. Using this equality and the relation $h_i a_i = a_1 x_1 + a_2 x_2 + a_3 x_3$, where a_i and h_i are lengths of sides and corresponding altitudes, we easily obtain a proof different from that of Eves. If $a_1 \leqq a_2 \leqq a_3$, we have at once $h_3 \leqq \rho + r \leqq h_1$. Returning to the function $f(B, C)$ there are two angles C_1, C_2 for which $f(44°, C) = 0$ which are approximately 95° 45.8′, 128° 14.2′, and $f(44°, C)$ is positive or negative according as C lies within or outside the interval from C_1 to C_2.