$$(7) \qquad \sum_{j=l+1}^{l+k} \lambda_j > m(v_1 + v_2 + \cdots + v_r - l + v_{r+1} + \cdots + v_i) + 2 = mk + 2.$$

If λ_l falls at the end of (v_r), then (5) drops out and we have again a contradiction. If $l=0$, the sum of the inequalities in (4) gives at once the contradiction that the sum of the k integers λ_j is $\geqq mk+2$, $i\geqq 2$.

Editorial Note. This interesting problem and its interesting solution were received from R. D. James with the following account of the origin of the problem. In an article by Heilbronn, Landau, and Sherk in the *Journal Tchécoslovaque de Mathématique et de Physique*, 65, 1935–36, pp. 117–140, there is a lemma (*Satz* 8) equivalent to the following: Given the numbers A_k^i defined as in (1) above but for $m=2$, then $A_k^{2k+1} \leqq k^{2k}$. After a study of this it seemed to James that the result should be an actual equality, but he could not find a proof and suggested the problem to J. S. Vigder. The latter considered the more general problem using the positive integer m in place of 2, and saw that a proof involved the polynomial theorem as above, but he was unable to complete the proof and passed the matter on to the proposer. The proposer formulated the problem differently and came through with a solution resulting from his proof of his lemma (3).

An Oval and its Normal Expansion

4036 [1942, 340]. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C_1 be an oval with a continuously varying radius of curvature R; at each point of C_1 a normal of length R is drawn exteriorly giving points of a second curve C_2 (which may not be convex); and let A be the area enclosed between the two curves. From a chosen fixed point a vector is drawn parallel to the normal at a point of C_1 and of length R for that point, thus giving as the point varies on C_1 a curve C_3 having the area A_3 and length L_3. If L_2 is the length of C_2 and A_1 is the area of C_1, show that

$$\text{(a)} \quad A = 3A_3; \qquad \text{(b)} \quad L_2L_3 \geqq 8\pi A_1;$$

the equality in (b) is true only when C_1 is a circle.

I. *Solution by Fritz John, University of Kentucky.* Let $p(\alpha)$ denote the "function of support" of C_1, *i.e.*, $p(\alpha)$ shall be the distance of that tangent of C_1 from the origin, whose normal forms the angle α with the x-axis (See Courant: *Calculus*, II, p. 213). Then

$$x = p\cos\alpha - p'\sin\alpha, \qquad y = p\sin\alpha + p'\cos\alpha$$

is a parametric representation for C_1. The radius of curvature of C_1 is given by $R=p+p''$, the enclosed area by

$$A_1 = \frac{1}{2}\int_0^{2\pi} (xy' - yx')d\alpha = \frac{1}{2}\int_0^{2\pi} (p^2 + pp'')d\alpha = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)d\alpha.$$

Similarly the parametric representations of C_2 and C_3 are respectively

$$x = (2p + p'') \cos \alpha - p' \sin \alpha, \qquad y = (2p + p'') \sin \alpha + p' \cos \alpha,$$

and

$$x = (p + p'') \cos \alpha, \qquad y = (p + p'') \sin \alpha;$$

hence the areas enclosed by C_2 and C_3 are easily found to be

$$A_2 = \frac{1}{2}\int_0^{2\pi} (4p^2 - 7p'^2 + 3p''^2)d\alpha$$

$$A_3 = \frac{1}{2}\int_0^{2\pi} (p^2 - 2p'^2 + p''^2)d\alpha.$$

Consequently $A = A_2 - A_1 = 3A_3$, which is the first statement.

Now "Wirtinger's inequality" states, that for a function $f(\alpha)$ of class C' with $\int_0^{2\pi} f(\alpha)d\alpha = 0$

$$\int_0^{2\pi} f'^2(\alpha)d\alpha > \int_0^{2\pi} f^2(\alpha)d\alpha,$$

unless f is of the form $f(\alpha) = a \cos \alpha + b \sin \alpha$; (see Hardy-Littlewood-Polya: *Inequalities*, pp. 185–187). For $f = p'$ it follows that $\int_0^{2\pi} p''^2(\alpha)d\alpha > \int_0^{2\pi} p'^2(\alpha)d\alpha$, and hence $A_3 > A_1$, unless $p' = a \cos \alpha + b \sin \alpha$; in the latter case $p = c + a \sin \alpha - b \cos \alpha$, and C_1 is a circle of radius c. The isoperimetric inequality (which may be based on Wirtinger's inequality), yields

$$L_2^2 \geqq 4\pi A_2, \qquad L_3^2 \geqq 4\pi A_3;$$

hence

$$L_2 L_3 \geqq 4\pi\sqrt{A_2 A_3} = 4\pi\sqrt{(3A_3 + A_1)A_3} > 4\pi\sqrt{4A_1^2} = 8\pi A_1$$

unless C_1 is a circle.

In the case where C_1 is a circle of radius c, C_2 is a circle of radius $2c$, and C_3 a circle of radius c, so that $L_2 L_3 = 8\pi A_1$.

II. *Solution by the Proposer*. We consider two normals to C_1 corresponding to the directions ϕ and $\phi + d\phi$; a point on a normal whose distance to C_1 is a constant equal to λ will describe a curve whose arc s^* satisfies

$$ds^* = (R + \lambda)d\phi.$$

The area A will be then

$$A = \iint ds^* d\lambda = \int_0^{2\pi} d\phi \int_0^R (R + \lambda)d\lambda = \frac{3}{2}\int_0^{2\pi} R^2 d\phi = 3A_3$$

which proves (a).

We have also, if s_2 is the arc of C_2,

(1) $$ds_2 = \sqrt{4R^2d\phi^2 + dR^2} = \sqrt{4R^2 + R'^2}\, d\phi$$

where R' represents the derivative with respect to ϕ. We have also

(2) $$ds_3 = \sqrt{R^2 + R'^2}\, d\phi.$$

From (1) and (2) we deduce, representing by s_1 the arc of C_1

$$ds_2 \geqq 2Rd\phi = 2ds_1 \quad \text{and} \quad L_2 \geqq 2L_1$$
$$ds_3 \geqq Rd\phi = ds_1 \quad \text{and} \quad L_3 \geqq L_1.$$

This gives us

(3) $$L_2L_3 \geqq 2L_1^2$$

But it is known that for every plane closed curve we have $L_1^2 - 4\pi A_1 \geqq 0$; so this inequality and (3) proves the last part (b).

The equality in (b) is valid only if $R' = 0$, and then the radius of curvature is constant and the closed curve must be a circle.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. H. F. Bright of San Angelo College has been appointed to an assistant professorship at Denison University.

Dr. Jesse Douglas has been appointed to an assistant professorship at Brooklyn College.

Associate Professor R. C. Hildner of Mt. Union College has been appointed to an assistant professorship at the College of Wooster.

Professor E. J. Moulton of Northwestern University is on leave of absence and Professor H. T. Davis is acting head of the department of mathematics.

Assistant Professor W. H. Myers has been appointed acting head of the mathematics department at San José State College.

Mr. N. D. Nelson of the University of Wisconsin has been appointed to an assistant professorship at Amherst College.

Dr. E. A. Nordhaus of the University of Wisconsin has been appointed to an assistant professorship at Michigan State College.

Assistant Professor C. V. L. Smith of Lafayette College is now a lieutenant,