The Kinematic Formula in Integral Geometry for Cylinders.

L. A. SANTALÓ (Buenos Aires, Argentina)

Summary. — We generalize the kinematic formula of Chern-Federer (1.2) to the case in which the moving manifold \( M' \) is a cylinder in \( E^n \). These cylinders and the corresponding kinematic density are suitably defined and some particular cases are considered in detail.

1. — Introduction.

This paper will be concerned with the so called «kinematic formula» in Integral Geometry, due to Federer [2] and Chern [1]. We shall refer mainly to the work of Chern, which likely assumes some more restrictive conditions than Federer, but remains into the mark of differential geometry. The approach of Federer is more in the mark of measure theory. The formula to which we refer is the following (Chern [1]).

Let \( M^s, M^t \) be a pair of orientable, compact, differentiable manifolds (without boundary) of dimensions \( p, q \) immersed in euclidean space \( E^n \). Let \( dg \) denote the kinematic density (= Haar measure of the group of motions in \( E^n \)) so normalized that the measure of all positions about a point is \( O_n, O_{n-1}, \ldots, O_1 \) where

\[
O_i = \frac{2\pi^{(i-1)/2}}{\Gamma((i+1)/2)}
\]

is the volume of the \( i \)-dimensional unit sphere. Assume \( M^s \) fixed and \( M^t \) moving with the kinematic density \( dg \). Let \( \mu_i(X^s) \) (\( 0 < e < k \)) be the integral invariants (we call them Weyl's curvatures) of the riemannian \( k \)-dimensional manifold \( X^s \) to be defined below. Then the kinematic formula of Chern-Federer writes

\[
\int \mu_i(M^s \cap gM^t) \, dg = \sum_{0 \leq i \leq e} c_i \mu_i(M^s) \mu_{e-i}(M^t), \quad i \text{ even}
\]

where \( e \) even and \( 0 < e < p + q - n \). The integral on the left is over the whole

\((*)\) Entrata in Redazione il 22 maggio 1973.
group of euclidean motions in $E^r$, i.e. over all positions of $M'$, and $c_i = c_i(n, p, q, e)$ are numerical constants depending on $n, p, q, e$ which may be calculated as follows. Put

$$c_{e-1} = \frac{O_{p+1} \cdots O_{p+2+q} O_{e+1} \cdots O_{e+1+e}}{O_{p+1} O_{p+2} O_{e+1} O_{e+2} O_{e+1+1} \cdots O_{e+1+e+1}} b_{e+p+q+e+1}$$

where the $b$'s are given by the following identity (with respect to the indeterminate $x$)

$$b_{e,m-1} = \sum (\frac{a}{2})! \frac{(2\lambda)!}{(2\lambda)! (e/2 - 2\lambda - \mu)!} O_{2\lambda} O_{2\lambda+e} \cdots x^{m-e-1} + \cdots + b_{e,m-1} x^{m-1}$$

where the sum on the left side is over the following range of indices

$$0 < 2\lambda + \mu < e/2, \quad 0 < \lambda, \mu.$$

For instance, for $i = 0$, (1.3) gives

$$c_0 = \frac{O_{e} \cdots O_{e+e} \cdots O_{e+1} \cdots O_{e+1+e}}{O_{e} O_{e+1} O_{e+2} O_{e+1+1} b_{e+p+q+e+1}}$$

and identifying the coefficients of $x^{e-1}$ of both sides of (1.4) we have (since the relations $0 < 2\lambda + \mu < e/2, 2\lambda + 2\mu = e$ give $\lambda = 0, \mu = e/2$)

$$b_{e,m-1} = \frac{O_{e-1} O_{e-2} O_{e-3}}{O_{e-1}}.$$

Hence

$$b_{e,p+q+e+1} = \frac{O_{p+1} \cdots O_{p+e} \cdots O_{p+1+1} \cdots O_{p+e+1+e}}{O_{p+1} O_{p+2} O_{p+1+1} b_{e,p+q+e+1}},$$

and thus

$$c_0 = \frac{O_{e} \cdots O_{e+e} \cdots O_{e+1} \cdots O_{e+1+e}}{O_{e} O_{e+1} O_{e+2} O_{e+1+1} b_{e+1+1}}.$$

In particular we have

$$c_0(n, p, q, 0) = \frac{O_{a} \cdots O_{i} O_{p+e} \cdots O_{e+e} \cdots O_{e+e}}{O_{e} O_{e+1} O_{e+2} O_{e+1+1} b_{e+1+1}}.$$
Our purpose is to extend (1.2) to the case in which \( M^* \) is a cylinder \( Z^* \) in \( E^* \). In this case, the kinematic density must be replaced by the density \( dZ_{k,m} \) for cylinders, which we will define in section 3. The result is the formula (5.2) which contains as special or limiting cases many formulas in integral geometry in \( E^n \). We consider with detail some of these particular formulas in section 6.

2. - The Weyl's curvatures.

We will define the curvatures \( \mu_s(X^*) \) which appear in the kinematic formula (1.2) (see WEYL [6], CHERN [1], FEDERER [2]).

Let \( X^* \) be a differentiable riemannian manifold of dimension \( k \) and consider the classical differential forms \( \omega_{\alpha}, \omega_{\beta} \) \((1 < \alpha, \beta, \gamma, \delta < k)\) of the moving frames method, such that

\[
\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 \quad d\omega_{\alpha} = \sum_{\beta} \omega_{\alpha\beta} \wedge \omega_{\beta} \quad d\omega_{\alpha\beta} = \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}
\]

where

\[
\Omega_{\alpha\beta} = \frac{1}{2} \sum_{\gamma, \delta} S_{\alpha\beta\gamma\delta} \omega_{\gamma} \wedge \omega_{\delta}
\]

The coefficients \( S_{\alpha\beta\gamma\delta} \) are essentially (though not exactly) the components of the Riemann-Christoffel tensor and have the same well known symmetry properties

\[
\left\{
\begin{array}{c}
S_{\alpha\beta\gamma\delta} = - S_{\alpha\beta\delta\gamma} = - S_{\beta\alpha\gamma\delta} \\
S_{\alpha\beta\gamma\delta} = S_{\gamma\delta\alpha\beta} , \quad S_{\alpha\beta\gamma\delta} + S_{\alpha\gamma\delta\beta} + S_{\alpha\delta\beta\gamma} = 0
\end{array}
\right.
\]

Put

\[
I_s = \frac{(-1)^{s(k-s)}!}{2^{s}k!} \sum_{\alpha, \beta, \gamma, \delta} \delta_{\alpha, \beta, \gamma, \delta}^s S_{\alpha\beta\gamma\delta} S_{\alpha\beta\gamma\delta} \ldots S_{\alpha\beta\gamma\delta}
\]

where \( s \) is an even integer satisfying \( 0 < s < k \) and \( \delta_{\alpha, \beta, \gamma, \delta}^s \) is equal to \(+1\) or \(-1\) according as \( \alpha, \ldots, \alpha \) is an even or odd permutation of \( \beta, \ldots, \beta \) and is otherwise zero, and the summation is taken over all \( \alpha, \ldots, \alpha \) and \( \beta, \ldots, \beta \) independently from 1 to \( k \). When \( X^* \) is oriented and compact, we let

\[
\mu_s(X^*) = \int_{X^*} I_s \, d\sigma_s
\]

where \( d\sigma_s \) is the volume element. This formula (2.5) defines the Weyl's curvatures \((s \text{ even, } 0 < s < k)\). In particular we have

\[
\mu_s(X^*) = \text{total volume of } X^*
\]
and, if \( k \) is even,

\[
\mu_\varepsilon(X^\varepsilon) = \frac{1}{2} \, O_\varepsilon \chi(X^\varepsilon)
\]

where \( \chi(X^\varepsilon) \) denotes the Euler-Poincaré characteristic of \( X^\varepsilon \). (2.7) is the Gauss-Bonnet formula for compact even dimensional manifolds.

It would be of interest to compare these curvatures \( \mu_\varepsilon(X^\varepsilon) \) with other curvatures which appear in the literature. For instance, if \( X^\varepsilon \) is the boundary of a bounded convex set of \( E^{n+1} \) the volume \( V(\varepsilon) \) of the parallel set to \( X^\varepsilon \) at distance \( \varepsilon \) is (HADWIGER [3])

\[
V(\varepsilon) = \sum_{i=0}^{k+1} \binom{k+1}{i} W_i \varepsilon^i
\]

and the volume of the \( \varepsilon \) tube at distance \( \varepsilon \) is

\[
V(\varepsilon) - V(-\varepsilon) = 2 \sum_{i=0}^{k+1} \binom{k+1}{i} W_i \varepsilon^i, \quad \text{if } i \text{ odd}.
\]

The invariants \( W_i(X^\varepsilon) \) (quermassintegrals, introduced by Minkowski) may be written

\[
W_i = \frac{1}{k+1} M_{i-1}
\]

where \( M_i \) (\( i = 0, 1, 2, \ldots, k \)) are the \( i \)-th integrated mean curvatures

\[
M_i = \frac{1}{k+1} \int_{X^\varepsilon} \left[ \frac{1}{R_1} \ldots \frac{1}{R_n} \right] d\sigma_{x}
\]

where \( d\sigma_{x} \) is the volume element of \( X^\varepsilon \) and \( \{1/R_1, \ldots, 1/R_n\} \) is the \( i \)-th elementary symmetric function of the principal curvatures of \( X^\varepsilon \). Comparing (2.9) with the Weyl's formula for the volume of tubes [6], we get

\[
\mu_\varepsilon = M_\varepsilon, \quad \varepsilon \text{ even}.
\]

This formula holds for smooth compact hypersurfaces \( X^\varepsilon \) of \( E^{n+1} \), not necessarily convex. We deduce that, for \( \varepsilon \) even, the mean curvatures (2.11) are isometric invariants of \( X^\varepsilon \) which do not depend of its immersion in \( E^{n+1} \), i.e. are intrinsic invariants.

3. - Density for cylinders.

Let \( M^a \) be an orientable, compact, differentiable manifold (without boundary) which belongs to a \((n-m)\)-dimensional linear space \( E^{n-m} \) in \( E^n \). Thus

\[
h + m < n.
\]
Through each point \( x \in M^k \) we consider the \( m \)-dimensional linear space \( E^m \) perpendicular to \( E^{k-m} \). The set of all these \( E^m \) is called a cylinder \( Z_{k,m} \) of dimension \( k + m \), whose generators (or generating \( m \)-spaces) are the \( m \)-spaces \( E^m \) and which orthogonal cross section is the manifold \( M^k \).

If we assume \( Z_{k,m} \) moving in \( E^n \), its position may determined by a \( E^{k-m}(0) \) through a fixed point 0, orthogonal to the generators \( E^m \), and the position of the cross section \( M^k \) in \( E^{k-m}(0) \). The density \( dE^{k-m}(0) \) (volume element of the grassmann manifold \( G_{n-m,n} \) of all \( (n-m) \)-planes through 0 in \( E^n \)) and \( dg_{k-m} \) (kinematic density in \( E^{k-m} \)) are well known (see, for instance Santaló [4], [5], Chern [1], Hadwiger [3]). The density \( dZ_{k,m} \) for cylinders \( Z_{k,m} \) is then

\[
dZ_{k,m} = dE^{k-m}(0) \wedge dg_{k-m}.
\]

We recall these densities for completeness. If \( (x; e_1, e_2, ..., e_n) \) is an orthogonal frame in \( E^n \) and we put

\[
\omega_i = (dx \cdot e_i), \quad \omega_{in} = de_i \cdot e_n
\]

then

\[
dg_n = \Lambda \omega_1 \Lambda \omega_2
\]

where the exterior products are between the ranges

\[
i = 1, 2, ..., n; \quad j = 2, 3, ..., n; \quad k = 1, 2, ..., n - 1
\]

with

\[
j > k.
\]

The differential form

\[
d\sigma_n = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n
\]

is the volume element in \( E^n \). If \( E^{k-m} \) is spanned by \( e_{m+1}, e_{m+2}, ..., e_n \) we have

\[
dg_{n-m} = \omega_{m+1} \wedge ... \wedge \omega_n \Lambda \omega_jk
\]

where

\[
j = m + 2, ..., n; \quad k = m + 1, ..., n - 1, \quad k < j
\]

and if \( E^m \) is spanned by \( e_1, e_2, ..., e_m \) we have

\[
dg_m = \omega_1 \wedge ... \wedge \omega_m \Lambda \omega_jk
\]

where

\[
j = 2, 3, ..., m; \quad k = 1, 2, ..., m - 1, \quad k < j.
\]
Finally, assuming $E^{s-m}(0)$ parallel to $E^{s-m}$, we have

$$dE^{s-m}(0) = A\omega_N$$

where

$$j = m + 1, \ldots, n, \quad k = 1, 2, \ldots, m,$$

From (3.4), (3.2), (3.7) and (3.8) we deduce

$$dg = dg_\ast = dE^{s-m}(0) \wedge dg_{s-m} \wedge dg_m = dZ_{s-m} \wedge dg_m.$$

The exterior products in (3.8) and (3.7) have a clear geometrical meaning. Indeed we have

$$\omega_1 \wedge \cdots \wedge \omega_m = d\sigma_m = \text{volume element in } E^m,$$

$$\omega_{m+1} \wedge \cdots \wedge \omega_n = d\sigma_{s-m} = \text{volume element in } E^{s-m}.$$  

$$(3.13) \quad A\omega_N(j = 2, \ldots, m; k = 1, \ldots, m - 1; k < j) = dO_{m-1} \wedge dO_{m-1} \wedge \cdots \wedge dO_1$$

where $dO_i$ denotes the area element on the unit $i$-dimensional sphere in the space spanned by $e_{i}, e_{i+1}, \ldots, e_{i+1}$, and

$$(3.14) \quad A\omega_N(j = m + 2, \ldots, n; k = m + 1, \ldots, n - 1; k < j) = dO_{s-m-1} \wedge \cdots \wedge dO_1$$

where $dO_i$ is now the area element on the unit sphere in the $(i+1)$-space spanned by $e_{m+1}, \ldots, e_{m+1}$.

The density $dE^m$ for the generating $m$-spaces $E^m$ writes

$$dE^m = dE^{s-m}(0) \wedge d\sigma_{s-m}$$

where $d\sigma_{s-m}$ is the element of volume in $E^{s-m}(0)$ at the intersection point $E^m \cap E^{s-m}(0)$.

Having into account (3.2) and (3.7) we have

$$dZ_{s,m} = dE^m \wedge dO_{s-m-1} \wedge \cdots \wedge dO_1.$$

We always consider the densities in absolute value, so that the order of the exterior products above is immaterial.

4. The Weyl’s curvatures for cylinders.

Choose the frame $(x; e_1, e_2, \ldots, e_n)$ such that $x \in Z_{s,m}$ and $e_1, e_2, \ldots, e_{m+k}$ span the tangent space to $Z_{s,m}$ in such a way that $e_1, e_2, \ldots, e_m$ span the generator $E^m$. 

through \( x \) and \( e_{m+1}, \ldots, e_{m+h} \) span the tangent space to the cross section \( M^a \). The volume elements in \( E^a, M^a \) and \( Z_{k,m} \) are, respectively

\[
(4.1) \quad d\sigma_a = \omega_1 \wedge \ldots \wedge \omega_m, \quad d\sigma_k = \omega_{m+1} \wedge \ldots \wedge \omega_{m+k}, \quad d\sigma_{m+k} = \omega_1 \wedge \ldots \wedge \omega_{m+h}
\]

and we have

\[
(4.2) \quad d\sigma_{m+k} = d\sigma_m \wedge d\sigma_k.
\]

Since all \( E^a \) are perpendicular to \( E^{a-m} \), we have \( e_a = \text{constant} \) for \( a = 1, 2, \ldots, m \) and thus \( \omega_{ab} = d\sigma_a \cdot e_b = 0 \) \( (a = 1, \ldots, m; k = 1, 2, \ldots, m+h) \). Therefore \( d\omega_{ab} = \sum \omega_{en} \wedge \omega_{en} + Q_{as} = 0 \) and consequently \( Q_{as} = 0 \). Therefore, applying (2.2) to the cylinder \( Z_{k,m} \) we have \( S_{asij} = 0 \) \( (a = 1, 2, \ldots, m; k, i, j = 1, 2, \ldots, m+h) \). According to the symmetry properties (2.3) of \( S_{asij} \) the equation \( S_{asij} = 0 \) implies that \( S_{asij} = S_{asij} = S_{asij} = 0 \). The remaining \( S_{asij} \) with \( i, j, k, s = m+1, m+2, \ldots, m+h \) are the functions \( S_{asij} \) corresponding to the cross section \( M^a \).

Therefore, the sums on the right side of (2.4) are the same for \( Z_{k,m} \) and for \( M^a \) and an easy calculation gives

\[
(4.3) \quad I_\sigma(Z_{k,m}) = \begin{cases} \left( \frac{h}{e^2} \right) I_\sigma(M^a) & \text{if } e < h, \\ \left( \frac{h+m}{e^2} \right) I_\sigma(M^a) & \text{if } e \geq h. \end{cases}
\]

5. - The kinematic formula for cylinders.

Let \( E^a \) be a generator of \( Z_{k,m} \) and consider a bounded domain \( D^a \subset E^a \). Assume that \( D^a = D^a(t) \) depends on a parameter \( t \) in such a way that \( D^a \to E^a \) when \( t \to \infty \). Consider the compact manifold \( D^a \times M^a \). If \( d\sigma_a, d\sigma_k, d\sigma_{k,m} \) denote respectively the volume elements in \( D^a(t), M^a \) and \( Z_{k,m} \) we have \( d\sigma_{k,m} = d\sigma_m \wedge d\sigma_k \) and from (2.5) and (4.3) we get

\[
(5.1) \quad \mu_\sigma(Z_{k,m}) = \begin{cases} \left( \frac{h}{e^2} \right) \mu_\sigma(M^a) \sigma_m & \text{if } e < h, \\ \left( \frac{h+m}{e^2} \right) \mu_\sigma(M^a) \sigma_m & \text{if } e \geq h. \end{cases}
\]

where \( \sigma_m \) denotes the volume of \( D^a \).

We now apply formula (1.2) to \( M^a \) and \( M^e = D^a \times M^a \). Using (3.10) having into account that \( dg_m = d\sigma_m \wedge dO_{m-1} \wedge \ldots \wedge dO_1 \) and making \( t \to \infty \) (after division of
both sides by $\sigma_m$, so that $D^m \rightarrow E^m$ we get the desired formula

$$\int_{Z_{h,m} \cap M^* \cap E^m} \mu(M^* \cap Z_{h,m}) dZ_{h,m} = \sum_{e-h-i \leq n} \frac{c_i}{O_{1}O_{2} \cdots O_{n-1}} \begin{pmatrix} h \cr e-i \end{pmatrix} \mu_i(M^*)\mu_{e-i}(M^*)$$

$$\mu_{e-i}=0 \text{ for } e>i.$$ (\text{even, } 0<e<p+h+m-n, \ i \geq 0, \ i \text{ even})

where $c_i = c_i(n, p, h+m, e)$ are the same constants as in Chern's formula (1.2).

6. - Particular cases.

1) Assume that $Z_{h,m}$ reduces to a $m$-plane $E^m$ ($h=0$). Then, according to (3.16) we have $dZ_{h,m} = dE^m \wedge dO_{n-m-1} \wedge \cdots \wedge dO_1$ and $\mu_0(M^*) = 1$. The sum on the right side of (5.2) reduces to the term $i = e$ and according to (1.6) we get ($e$ even, $e<p+h+m-n$)

$$\int_{E^m \cap M^* \cap E^m} \chi(M^* \cap E^m) dE^m = \frac{O_{n-m} \cdots O_{n}O_{p+1}O_{p+2} \cdots O_{p+m-n+1}}{O_{1}O_{2} \cdots O_{n-1}} \mu(M^*)$$

This formula is due to Chern [1]. For $p=n-1$, see [4], [5].

2) Consider the case $e=p+m-n$. According to (2.7) we have

$$\mu_{p+m-n}(M^* \cap E^m) = \frac{1}{2} \mu(M^* \cap E^m)$$

and (6.1) gives ($p+m-n$ even)

$$\int_{E^m \cap M^* \cap E^m} \chi(M^* \cap E^m) dE^m = \frac{2O_{n-m} \cdots O_{n}O_{p+1}O_{p+2} \cdots O_{p+m-n+1}}{O_{1}O_{2} \cdots O_{n-1}} \mu_{p+m-n}(M^*)$$

3) The case $e=0$. Applying (1.7), from (5.2) we deduce

$$\int_{Z_{h,m} \cap M^* \cap E^m} \mu_0(M^* \cap Z_{h,m}) dZ_{h,m} = \frac{O_{n-m} \cdots O_{n}O_{p+1}O_{p+2} \cdots O_{p+m-n+1}}{O_{1}O_{2} \cdots O_{n-1}} \mu_0(M^*)$$

If $p+h+m-n = 0$, then $\mu_0$ is equal to the number $v$ of intersection points of $M^*$ and $Z_{h,m}$, and (6.3) gives

$$\int_{Z_{h,m} \cap M^* \cap E^m} \nu(M^* \cap Z_{h,m}) dZ_{h,m} = \frac{2O_{n-m} \cdots O_{n}O_{p+1}O_{p+2} \cdots O_{p+m-n+1}}{O_{1}O_{2} \cdots O_{n-1}} \mu_0(M^*)$$
and (6.1) gives

\[ (6.5) \quad \int v(M^s \cap E^n) \, dE^n = \frac{2\mu_{n-m} \ldots \mu_n}{\mu_{n-1} \ldots \mu_n} \mu_{n-1}(M^s). \]

The integrals in (6.4) and (6.5) are extended over all positions of \( Z_{k,m} \) and \( E^n \), \( \nu \) being zero if they do not intersect the manifold \( M^s \).

4) As last example, consider the case

\[ p + h + m - n = 2, \quad e = 2. \]

We have, by (2.7)

\[ \mu_{e}(M^s \cap Z_{k,m}) = \frac{1}{2} O_{e} \chi(M^s \cap Z_{k,m}) \]

From (1.3), (1.4), having into account that \( q = h + m, \ p + q - n = 2 \), we have

\[ (6.6) \quad c_{e}(n, p, h + m, 2) = \frac{O_{e} \ldots O_{s-1}O_{s}O_{e}}{O_{e}O_{p+1}O_{k+m}O_{1}}. \]

Using (6.6), (1.8) and the identity \( O_{e}O_{e-1} = (e-1)O_{e} \), formula (5.2) gives

\[ (6.7) \quad \int_{M^s \cap Z_{k,m}} \chi(M^s \cap Z_{k,m}) \, dZ_{k,m} = \frac{2\mu_{n} \ldots \mu_{s}}{O_{s}O_{k+m}O_{1}} \cdot \left[ \frac{h(h-1)}{2\pi(h+m-1)} \mu_{e}(M^s)\mu_{s}(M^s) + \frac{O_{s-1}}{O_{p+1}} \mu_{e}(M^s)\mu_{s}(M^s) \right]. \]

REFERENCES