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CAUCHY AND KUBOTA's FORMULA FOR CONVEX BODIES IN ELLIPTIC n-SPACE

Summary: *The classical integral formulas de CAUCHY and KUBOTA of the theory of convex bodies are extended to elliptic n-dimensional space. The CAUCHY's formula takes the form (4.7) and that of KUBOTA takes the form (5.5).*

1. Introduction

Let K_n be a compact convex body in elliptic n-dimensional space S^n. Let L_r denote an r-plane (r-dimensional subspace) in S^n which do not intersect K_n and let L^*_{n-r-1} denote the $(n-r-1)$-plane dual of L_r. The $(r+1)$-planes $L_{r+1}[L_r]$ through L_r which meet K_n determine in L^*_{n-r-1} a convex set K^*_{n-r-1} (projection of K_n into L^*_{n-r-1} from L_r). Let V^*_{n-r-1} denote the $(n-r-1)$-dimensional volume of K^*_{n-r-1} and M_i^* $(i=0,1,\dots,n-r-2)$ the i-th integrals of mean curvature of ∂K^*_{n-r-1}. In this paper we shall obtain some integral formulas referring to V^*_{n-r-1} and M_i^* from which, in particular, one can deduce the mean values of these magnitudes with respect to all r-planes L_r exterior to K_n. These mean values generalize to elliptic space the classical formulas of Cauchy and Kubota for convex bodies in euclidean n-space (see [1], pp. 48-49; [3], pp. 217-218).

For simplicity, we shall assume that ∂K_n is of class C^2 in order that the integrals of mean curvature be well defined. However, using the relation $M_i(\partial K_n)=nW_{i+1}(K_n)$ ([3], p. 224) and applying the theorem that any convex hypersurface can be approximated by a sequence of analytic convex hypersurfaces, it follows that the obtained integral formulas can be expressed

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in terms of the quermassintegrale W_i and therefore they hold for general compact convex bodies in S^n.

2. Some known formulas

For notations and details we refer to [3]. Let dL_r denote the density, in the sense of integral geometry, for r-planes L_r in S^n ([3], p. 305). With this density, the total measure of all L_r in S^n $(0 \leqslant r \leqslant n-1)$ is given by ([3], p. 309),

$$\int_{\text{Total}} dL_r = \frac{0_n 0_{n-1} \ldots 0_{r+1}}{0_{n-r-1} \ldots 0_0} = \frac{0_n \ldots 0_{n-r}}{0_r \ldots 0_0}\,, \tag{2.1}$$

where $0_i = 2\pi^{(i+1)/2}/\Gamma((i+1)/2)$ denotes the surface area of the i-dimensional unit sphere. For $r=0$ (2.1) gives the volume of S^n, namely $0_n/2$, as is well known.

Let K_n be a compact convex body in S^n. If $M_i = M_i(\partial K_n)$ denotes the i-th integral of mean curvature of ∂K_n (for $i=0$, M_0 is the surface area of ∂K_n) and $M_i(\partial K_n \cap L_{r+1})$ denotes the i-th integral of mean curvature of the intersection of ∂K_n with a moving L_{r+1}, we have ([3], p. 248),

$$\int_{K_n \cap L_{r+1}} \neq \phi M_i(\partial K_n \cap L_{r+1})\, dL_{r+1} = \frac{0_{n-2} \ldots 0_{n-r-1}\, 0_{n-i}}{0_{r-1} \ldots 0_0\, 0_{r+1-i}} M_i(\partial K_n) \tag{2.2}$$

This formula holds without change in euclidean and elliptic spaces.

If $\sigma_{r+1}(K_n \cap L_{r+1})$ denotes the volume of the convex set $K_n \cap L_{r+1}$ we have ([3], p. 309)

$$\int_{K_n \cap L_{r+1}} \neq \phi \sigma_{r+1}(K_n \cap L_{r+1})\, dL_{r+1} = \frac{0_{n-1} \ldots 0_{n-r-1}}{0_r \ldots 0_0} V \tag{2.3}$$

where $V = V(K_n)$ denotes the volume of K_n.

Finally, let us recall that the measure of all r-planes L_r intersecting K_n is given by ([3], p. 310)

$$\int_{K_n \cap L_r \neq \phi} dL_r = \frac{0_{n-2} \ldots 0_{n-r}}{0_r \ldots 0_1} \left[0_{n-1} V + \sum_{i=1}^{r'} \binom{r-1}{2i-1} \frac{0_r 0_{r-1}\, 0_{n-2i+1}}{0_{2i-1}\, 0_{r-2i} 0_{r-2i+1}} M_{2i-1} \right] \tag{2.4}$$

for r even $(r = 2r')$, and

$$\int_{K_n \cap L_r \neq \phi} dL_r = \frac{0_{n-2} \ldots 0_{n-r}}{0_{r-1} \ldots 0_1} \sum_{i=0}^{r'} \binom{r-1}{2i} \frac{0_{r-1}\, 0_{n-2i}}{0_{2i} 0_{r-2i-1}\, 0_{r-2i}} M_{2i} \tag{2.5}$$

for r odd $(r=2r'+1)$.

For $r=1$ $(r'=0)$ (2.5) is not directly applicable. However, using the identity $(0_{n-2}\dots 0_{n-r})\,(0_{r-1}\dots 0_1)^{-1}=(0_{n-2}\dots 0_r)\,(0_{n-r-1}\dots 0_1)^{-1}$, we get (putting $M_0=F$),

$$\int_{K_n\cap L_1\neq\phi} dL_1=(0_n/4\pi)\,F \tag{2.6}$$

For $n\geqslant 3$, $r=2$, (2.4) gives

$$\int_{K_n\cap L_2\neq\phi} dL_2=(0_{n-2}\,0_{n-1}/8\pi^2)\,(V+M_1)\;,$$

3. Solid angles

Consider a compact convex body K_n and an r-plane L_r $(0\leqslant r\leqslant n-2)$ in S^n which do not intersect. The set of all $(r+1)$-planes L_{r+1} which contain L_r and meet K_n, can be measured with the invariant density $dL_{r+1}[L_r]$ ([3], p. 202) which has the same explicit form for the elliptic as for the euclidean space and coincides with the invariant element of volume of the Grassmann manifold $G_{1,n-r-1}$. We call this measure the *solid angle* which subtends K_n from L_r and we denote it by

$$\phi_r^{(n)}=\int dL_{r+1}[L_r] \tag{3.1}$$

where the integral is extended over all L_{r+1} which satisfy the following conditions

$$L_r\subset L_{r+1}\;,\qquad L_{r+1}\cap K_n\neq\phi\;.$$

Notice that for $r=0$, $\phi_0^{(n)}$ is the usual solid angle under which K_n is seen from the point L_0. For $r=n-2$, $\phi_{n-2}^{(n)}$ is the angle between the support hyperplanes to K_n through L_{n-2}.

According to the given definition (3.1), the solid angle $\phi_r^{(n)}$ is equal to the $(n-r-1)$-dimensional volume V_{n-r-1}^* of the convex set K_{n-r-1}^* dual of K_n defined in the introduction.

4. Integral formulas

We have denoted by $dL_{r+1}[L_r]$ the density for $(r+1)$-planes L_{r+1} about a fixed r-plane L_r. Calling now $dL_r(L_{r+1})$ the density for r-planes L_r in L_{r+1}, we have the following formula

$$dL_r(L_{r+1}) \wedge dL_{r+1} = dL_r \wedge dL_{r+1}[L_r] \tag{4.1}$$

which is essentially due to B. Petkantschin [2] and has the same form for elliptic as for euclidean space (see also [3], p. 207).

We want to compute the integral of both sides of (4.1) over all r- and $(r+1)$-planes such that

$$L_r \cap K_n = \phi \quad , \quad L_r \subset L_{r+1} \quad , \quad L_{r+1} \cap K_n \neq \phi \, .$$

According to (3.1) the right side member gives

$$\int_{K_n \cap L_r = \phi} \phi_r^{(n)} dL_r \, ,$$

In order to compute the integral of the differential form on the left of (4.1), we first leave L_{r+1} fixed and observe that the integral of all L_r such that $L_r \subset L_{r+1}$ and $L_r \cap K_n = \phi$, is equal to the total measure of the r-planes in L_{r+1} (which, according to (2.1) is O_{r+1}/O_0), less the measure of the r-planes $L_r \subset L_{r+1}$ which meet the convex set $K_n \cap L_{r+1}$. Putting $M_i^{(r+1)} = M_i(\partial K_n \cap L_{r+1})$ and $V^{(r+1)} = V(K_n \cap L_{r+1})$ this measure, according to (2.4) and (2.5) is

a) For $r = 2r'$, $n = r+1 = 2r'+1$,

$$\int_{K_n \cap L_r \neq \phi} dL_r(L_{r+1}) = V^{(r+1)} + \sum_{i=1}^{r'} \binom{r-1}{2i-1} \frac{O_{r-1} O_{r+2-2i}}{O_{2i-1} O_{r-2i} O_{r-2i+1}} M_{2i-1}^{(r+1)} \tag{4.3}$$

b) for $r = 2r'+1$, $n = r+1 = 2(r'+1)$,

$$\int_{K_n \cap L_r \neq \phi} dL_r(L_{r+1}) = \sum_{i=0}^{r'} \binom{r-1}{2i} \frac{O_{r-1} O_{r+1-2i}}{O_{2i} O_{r-2i-1} O_{r-2i}} M_{2i}^{(r+1)} \tag{4.4}$$

Thus, multiplying the difference between O_{r+1}/O_0 and the measures (4.3) or (4.4) by dL_{r+1} and performing the integration over all L_{r+1} which meet K_n, having into account (2.2) and (2.3) and equating with (4.2) we get:

a) For $r = 2r'$ (r even),

$$\int_{K_n \cap L_r \neq \phi} \phi_r^{(n)} dL_r = \frac{O_{r+1} O_{n-2} \ldots O_{n-r-1}}{O_0 O_r \ldots O_1} \sum_{i=0}^{r'} \binom{r}{2i} \frac{O_r O_{n-2i}}{O_{2i} O_{r-2i} O_{r-2i+1}} M_{2i}$$

$$-\frac{0_{n-1}\dots 0_{n-r-1}}{0_r\dots 0_0}V-\sum_{i=1}^{r'}\binom{r-1}{2i-1}\frac{0_{r-1}\,0_{r+2-2i}}{0_{2i-1}\,0_{r-2i}0_{r-2i+1}}\;\frac{0_{n-2}\dots 0_{n-r-1}\,0_{n-2i+1}}{0_{r-1}\dots 0_0\,0_{r-2i+2}}M_{2i-1} \quad (4.5)$$

b) For $r=2r'+1$ (r odd),

$$\int_{K_n\cap L_r\neq\phi}\phi_r^{(n)}dL_r=\frac{0_{n-2}\dots 0_{n-r-1}}{0_0\,0_r\dots 0_1}\Bigg[0_{n-1}V+$$

$$\sum_{i=1}^{r'+1}\binom{r}{2i-1}\frac{0_{r+1}\,0_r\,0_{n-2i+1}}{0_{2i-1}\,0_{r+1-2i}0_{r+2-2i}}M_{2i-1}\Bigg] \quad (4.6)$$

$$-\sum_{i=0}^{r'}\binom{r-1}{2i}\frac{0_{r-1}\,0_{r+1-2i}}{0_{2i}0_{r-2i-1}\,0_{r-2i}}\;\frac{0_{n-2}\dots 0_{n-r-1}\,0_{n-2i}}{0_{r-1}\dots 0_0\,0_{r+1-2i}}M_{2i}$$

The case $r=0$. The preceding formulas are not directly applicable for $r=0$. In this case, we can proceed as follows. Let λ denote the length of the chord $K_n\cap L_1$. The integral of the left side of (4.1) gives $\int(\pi-\lambda)\,dL_1$ and since the measure of the set of lines L_1 which meet K_n is $(0_n/4\pi)\,F$ (according to (2.6)) and it is known that $\int\lambda\,dL_1=(0_{n-1}/2)\,V$ ([3], p. 307), we have

$$\int_{L_0\notin K_n}\phi_0^{(n)}dL_0=(0_n/4)\,F-(0_{n-1}/2)\,V\,. \quad (4.7)$$

In this formula the solid angle $\phi_0^{(n)}$ is equal to the volume of the projection of K_n into the hyperplane dual of L_0, so that (4.7) is the extension to the elliptic space of the so called formula of Cauchy for convex bodies in euclidean space ([1], p. 48; [3], p. 218).

As a consequence we have that the mean value of the volume of the projection $\phi_0^{(n)}$ is

$$E(\phi_0^{(n)})=\frac{0_nF-20_{n-1}V}{20_n-4V}$$

where V denotes the volume and F the surface area of K_n.

Some particular cases. The cases $n=2,3$ are known (see [3] p. 318-319 and [4] p. 186). They write

$$\int\phi_0^{(2)}dL_0=\pi(L-F)$$

$$\int\phi_0^{(3)}dL_0=(1/2)\,\pi^2F-2\pi\,V$$

$$\int \phi_1^{(3)} dL_1 = 2\pi (M_1 + V) - (1/2)\, \pi^2 F$$

For $n \geqslant 4$ the results are new. For $n = 4$ we get the following possibilities

$$\int \phi_0^{(4)} dL_0 = (2/3)\, \pi^2 F - \pi^2 V$$

$$\int \phi_1^{(4)} dL_1 = 2\pi^2 (V + M_1) - (4/3)\, \pi^2 F$$

$$\int \phi_2^{(4)} dL_2 = \pi^2 [(2/3)\, F + M_2 - M_1 - V]$$

In all cases, the integrals are extended over all L_0, L_1, L_2 exterior to the corresponding convex body.

5. More integral formulas

In the elliptic space, the principle of duality allows to assign to each integral formula its dual. Given a compact convex set K_n, the hypersurface parallel to ∂K_n in a distance $\pi/2$, is called the hypersurface "polar" of ∂K_n and it is the boundary of a convex body K_n^p (which does not contain K_n).The integrals of mean curvature of ∂K_n and ∂K_n^p satisfy the relation ([3], p. 304)

$$M_i(\partial K_n^p) = M_{n-1-i}(\partial K_n) \quad , \quad i = 0, 1, \ldots, n-1 \tag{5.1}$$

The formula (2.2) may be written

$$\int_{K_n \cap L_r \neq \phi} M_i^{(r)}(\partial K_n \cap L_r)\, dL_r = \frac{0_{n-2} \ldots 0_{n-r} 0_{n-i}}{0_{r-2} \ldots 0_0 0_{r-i}}\, M_i(\partial K_n) \tag{5.2}$$

which holds for $i = 0, 1, \ldots, r-1$; $r = 2, 3, \ldots, n-1$.

By duality, using (5.1), (5.2) transforms into

$$\int_{K_n \cap L_{n-r-1} = \phi} M_{r-1-i}^{(r)}(\partial K_r^*)\, dL_{n-r-1} = \frac{0_{n-2} \ldots 0_{n-r} 0_{n-r}}{0_{r-2} \ldots 0_0\, 0_{r-i}}\, M_{n-1-i}(\partial K_n) \tag{5.3}$$

where K_r^* denotes the projection of K_n from L_{n-r-1} into its dual r-plane.

By the change of indices $n-r-1 \to r$, $r-1-i \to i$ (5.3) takes the form

$$\int_{K_n \cap L_r = \phi} M_i^{(n-1-r)}(\partial K_{n-1-r}^*)\, dL_r = \frac{0_{n-2} \ldots 0_{r+1} 0_{r+i+2}}{0_{n-r-3} \ldots 0_0\, 0_{i+1}}\, M_{r+i+1}(\partial K_n) \tag{5.4}$$

which holds for $i=0,1,\dots,n-r-2$; $r=0,1,\dots,n-3$.

For $r=0$ we have

$$\int_{L_0 \notin K_n} M_i^{(n-1)}(\partial K_{n-1}^*)\, dL_0 = \frac{0_{n-2}\, 0_{i+2}}{2 0_{i+1}} M_{i+1}(\partial K_n) \tag{5.5}$$

which holds for $i=0,1,\dots,n-2$. This formula generalizes to elliptic space the so called formula of Kubota for convex bodies in euclidean space ([1], p. 49; [3], p. 217).

For $i=0$, denoting $F^*=M_0(\partial K_{n-1}^*)$ the surface area of K_{n-1}^*, we have

$$\int_{L_0 \notin K_n} F^*\, dL_0 = 0_{n-2} M_1 . \tag{5.6}$$

For $n=3$, if u denotes the perimeter of the projected set K_2^*, (5.6) writes

$$\int_{L_0 \notin K_3} u\, dL_0 = 2\pi M_1 , \tag{5.7}$$

The difference between the total measure (1.1) of all L_r in S^n and the measure (2.4) or (2.5) of those which meet K_n gives the measure of all L_r which are exterior to K_n and then (5.4) allows to write down the mean values of the integrals of mean curvature $M_i^{(n-1-r)}(\partial K_{n-1-r}^*)$. For instance, from (5.7) we deduce

$$E(u) = \frac{2\pi M_1}{\pi^2 - V} .$$

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