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AN INEQUALITY BETWEEN THE PARTS INTO WHICH A CONVEX BODY IS DIVIDED BY A PLANE SECTION

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A new proof is given of an inequality of J. Bokowski and E. Sperner [1] referring to the product of the volume of the two parts into which a convex body is divided by a plane. The proof, which is given for dimensions $n = 2, 3$ uses known formulas of Integral Geometry and is generalized to convex bodies of the elliptic and hyperbolic spaces.

1. Introduction.

Let K be a convex domain in the euclidean n-space E_n and let L_{n-1} be an hyperplane which divides K into two parts K_1 and K_2. Let $V(K_1)$, $V(K_2)$ denote the volumes of K_1 and K_2 respectively, D the diameter of K and σ_{n-1} the $(n-1)$-dimensional volume of the intersection $K \cap L_{n-1}$. Then, J. Bokowski and E. Sperner [1], [2] have proved the following inequality

$$V(K_1)\, V(K_2) \leq \frac{(1-2^{-n})\,(n-1)\,\omega_{n-1}}{n\,(n+1)}\, D^{n+1}\, \sigma_{n-1} \tag{1.1}$$

where ω_{n-1} denotes the volume of the $(n-1)$-dimensional unit sphere. For $n = 2, 3$ this inequality takes the form

$$F_1 F_2 \leq (D^3/4)\,\sigma_1, \qquad V_1 V_2 \leq (7/48)\,\pi D^4 \sigma_2. \tag{1.2}$$

Our purpose is to give a new proof of the particular cases (1.2) and to generalize these inequalities to the elliptic and hyperbolic spaces.

2. A fundamental Lemma.

Consider the segment OA on the real line, of length a, and the segment OX of length $x \leq a$. Let $f(r)$ be an integrable non-negative function defined on the closed interval $(0, a)$, which is strictly positive ($f(r) > 0$) for $0 < r < a$. Consider the integral

$$(2.1) \qquad I(x) = \int f(t_2 - t_1)\, dt_1 \wedge dt_2, \quad t_1 \in OX, \quad t_2 \in XA.$$

Then we have the following

LEMMA. *For any function $f(r)$ which satisfies the stated conditions, the integral* (2.1) *has its maximum for $x = a/2$.*

Proof. Let $F(r)$ be a primitive of $f(r)$, with $r = t_2 - t_1$, and $G(r)$ a primitive of $F(r)$. We have

$$(2.2) \qquad I(x) = \int_0^x [F(a - t_1) - F(x - t_1)]\, dt_1 = $$
$$= -G(a - x) + G(0) + G(a) - G(x).$$

In order that $I(x)$ have a maximum or minimum at the point x we have $I'(x) = F(a - x) - F(x) = 0$ and since $F(x)$ is an increasing function we will have $a - x = x$ and $x = a/2$. This critical value $I(a/2)$ is a maximum because $I(0) = I(a) = 0$.

3. The case $n = 2$.

We want to consider separately the cases of the euclidean, elliptic and hyperbolic planes.

a) *The euclidean plane.* Consider the line G_0 which divides K into two convex domains K_1 and K_2. Let σ_1 denote the length of the chord $G_0 \cap K$ (fig. 1). Consider the pair of points $P_1 \in K_1$, $P_2 \in K_2$ and the line G determined by

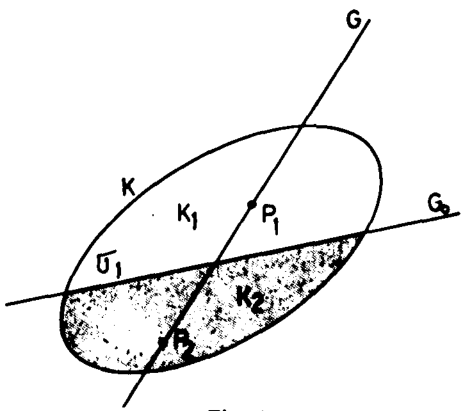

Fig. 1

them. It is well known the differential formula

$$d P_1 \wedge d P_2 = |t_2 - t_1| \, d G \wedge d t_1 \wedge d t_2 \tag{3.1}$$

where $d P_1$, $d P_2$ are the area elements of the plane at P_1, P_2, $d G$ is the density for lines on the plane and t_1, t_2 are the abscissas of P_1, P_2 on G [3, p. 28 and 46].

Integration of both sides of (3.1) over all pairs $P_1 \in K_1$, $P_2 \in K_2$ gives: on the left side we get $F_1 F_2$ and in the right side we have the integral (2.1) for the values

$$r = t_2 - t_1, \quad f = r, \quad F = (1/2)\, r^2, \quad G = (1/6)\, r^3. \tag{3.2}$$

Therefore, denoting by a the lenght of the chord $G \cap K$, we get

$$I(x) = (1/2)\, a\, x\, (a - x), \quad I(a/2) = a^3/8. \tag{3.3}$$

Since the measure of the set of lines which cut the chord $K \cap G_0$ is equal to $2\sigma_1$ and $a \leq D$ ($D =$ diameter of K), we have

$$F_1 F_2 = \int I(x)\, d G \leq (D^3/4)\, \sigma_1 \tag{3.4}$$

which is the first inequality (1.2)

b) *The elliptic case.* On the elliptic plane, instead of (3.1), we have [3, p. 316],

$$d P_1 \wedge d P_2 = \sin |t_2 - t_1| \, d G \wedge d t_1 \wedge d t_2. \tag{3.5}$$

We apply the fundamental lemma for the values

$$f = \sin r, \quad F = -\cos r, \quad G = -\sin r \tag{3.6}$$

and we have

$$I(x) = \sin (a - x) - \sin a + \sin x. \tag{3.7}$$

By integrating (3.5) over all pairs of points $P_1 \in K_1$, $P_2 \in K_2$ we get

$$F_1 F_2 = \int (\sin (a - x) - \sin a + \sin x)\, d G \leq \int (2 \sin (a/2) - \sin a)\, d G$$
$$= 4 \int \sin (a/2) \sin^2 (a/4)\, d G \leq 8 \sin (D/2) \sin^2 (D/4)\, \sigma_1 .$$

Therefore we have the following inequality

$$F_1 F_2 \leq 8 \sin (D/2) \sin^2(D/4) \sigma_1 . \tag{3.8}$$

We have applied that the measure of lines G which cut a segment of length σ_1 is equal to $2\sigma_1$, the same that in the euclidean case [3, p. 310].

c) *The hyperbolic plane.* In this case, instead of (3.1) we have [3, p. 316]

$$d P_1 \wedge d P_2 = \sinh |t_2 - t_1| \, d G \wedge d t_1 \wedge d t_2 . \tag{3.9}$$

In order to apply the lemma, we have

$$f = \sinh r, \qquad F = \cosh r, \qquad G = \sinh r \tag{3.10}$$

and therefore

$$\begin{aligned} I(x) &= - \sinh (a - x) + \sinh a - \sinh x, \\ I(a/2) &= 4 \sinh (a/2) \sinh^2 (a/4). \end{aligned} \tag{3.11}$$

Since the measure of lines which intersect a segment of length σ_1 is also $2\sigma_1$, [3, p. 310] we get the inequality

$$F_1 F_2 \leq 8 \sinh (D/2) \sinh^2 (D/4) \sigma_1 \tag{3.12}$$

which is the generalization to the hyperbolic plane of the first inequality of Bokowski-Sperner (1.2).

4. The case $n = 3$.

We cosider the three cases:

a) *Euclidean space.* With the customary notation we have [3, p. 237],

$$d P_1 \wedge d P_2 = (t_2 - t_1)^2 d G \wedge d t_1 \wedge d t_2 . \tag{4.1}$$

By integration over all pairs $P_1 \in K_1$, $P_2 \in K_2$, where K_1 and K_2 are now the bodies into which K is partitioned by the plane E_0, calling V_1 and V_2 the volumes of these bodies, we have

$$f = (t_2 - t_1)^2 = r^2 , \qquad F = (1/3) r^3, \qquad G = (1/12) r^4 \tag{4.2}$$

and

$$I(x) = (-1/12)(a-x)^4 + (1/12)a^4 - (1/12)x^4, \qquad I(a/2) = (7/96)a^4.$$

Since the measure of the set of lines which cut the plane domain $E_0 \cap K$ is $\pi\sigma_2$, where σ_2 denotes the surface area of $E_0 \cap \mathrm{K}$ [3, p. 233], we get

$$V_1 V_2 \leq (7/96)\pi D^4 \sigma_2 \tag{4.3}$$

which is better than the second inequality of (1.2).

b) *Elliptic space.* In this case we have [3, p. 316]

$$d P_1 \wedge d P_2 = \sin^2 (t_2 - t_1)\, d G \wedge d t_1 \wedge d t_2. \tag{4.4}$$

In order to apply the lemma, we have now

$$f = \sin^2 r, \quad F = (1/2)(r - \sin r \cos r), \quad G = (1/4)(r^2 - \sin^2 r) \tag{4.5}$$

and according to (2.2) we have

$$I(a/2) = (1/2)\sin^4 (a/2) + (1/8)(a^2 - \sin^2 a) \tag{4.6}$$

and since the measure of the lines which cut $E_0 \cap K$ is equal to $\pi\sigma_2$ [3, p. 310], we get

$$V_1 V_2 \leq (1/8)(4 \sin^4 (D/2) + D^2 - \sin^2 D)\pi\sigma_2 \tag{4.7}$$

which generalizes the inequality of Bokowski-Sperner to the elliptic space.

c) *Hyperbolic space.* In this case we have [3, p. 316]

$$d P_1 \wedge d P_2 = \sinh^2 (t_2 - t_1)\, d G \wedge d t_1 \wedge d t_2 \tag{4.8}$$

and therefore we have, with the notations of n. 2,

$$f = \sinh^2 r, \quad F = (1/2)(\sinh r \cosh - r), \quad G = (1/4)(\sinh^2 r - r^2) \tag{4.9}$$

and thus

$$\begin{aligned} I(x) &= (1/4)(-\sinh^2 (a-x) + (a-x)^2 + \sinh^2 a - a^2 - \sinh^2 x + x^2), \\ I(a/2) &= (1/2)\sinh^4 (a/2) + (1/8)(\sinh^2 a - a^2). \end{aligned} \tag{4.10}$$

Therefore, since the measure of the set of lines which intersect the set $E_0 \cap K$ is equal to $\pi \sigma_2$ [3, p. 310], we get

$$V_1 V_2 \leq (1/8)(4 \sinh^4 (D/2) + \sinh^2 D - D^2) \pi \sigma_2 \tag{4.11}$$

which is the generalization to the hyperoblic space of the second inequality (1.2).

5. A conjecture.

We have considered the case in which K is partitioned by a line (for $n = 2$) or by a plane (for $n = 3$). More general is the case of a partition of K into two sets K_1, K_2 not necessarily convex, separated by a curve (for $n = 2$) or by a surface (for $n = 3$). To apply the foregoing proof in this case we will need a lemma more general that the lemma stated in n. 2. We state it as the following conjecture:

Consider the closed interval $(0, a)$ on the real line, divided into $n + 1$ parts by the points $0 < a_1 < a_2 < \ldots < a_n < a$. Put $a_0 = 0$, $a_{n+1} = a$ and consider the sets of intervals

$$\begin{aligned} T &= \{(0, a_1), \quad (a_2, a_3), \quad (a_4, a_5), \ldots\} \\ T^* &= \{(a_1, a_2), \quad (a_3, a_4), \ldots\}. \end{aligned} \tag{5.1}$$

Consider the integral

$$I(a_1, a_2, \ldots, a_n, a) = \int_{\substack{t \in T \\ t^* \in T^*}} f(|t - t^*|)\, dt\, dt^*. \tag{5.2}$$

The conjecture is that this integral has a maximum for $n = 1$ and $a_1 = a/2$ for any integrable and non-negative function defined on the interval $(0, a)$. If it is not true, seek additional conditions for f.

In order to apply this conjecture to the generalization of the inequality (1.1) to the elliptic and the hyperbolic spaces it should be sufficient to prove it for the cases $f = \sin^n r$ and $f = \sinh^n r$.