Two applications of the integral geometry in affine and projective spaces.

To Prof. O. Varga on his 50. anniversary with cordial friendship.

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Introduction.

The integral geometry in projective space was initiated by VARGA [9] and continuated, together with the integral geometry in affine space, by the present author [4].

In this paper we give two applications of these concepts. First we consider the density for sets of pairs of parallel hyperplanes invariant with respect to the unimodular affine group. Then we evaluate the measure of all pairs of parallel hyperplanes which contain a given convex body K: the result is the integral (3.1) where $\Delta(\sigma)$ is the width of K corresponding to the direction σ. Consequently, J (3.2) is an unimodular affine invariant of K and we obtain the inequalities (3.9) which relate J with the volume V of K.

The second application concerns the density for sets of hyperquadrics invariant with respect to the projective group. We give the explicit forms (5.14), (5.16) and (5.19) of this density. For $n=2$ (conics on the plane) the formula (5.19) was given by STOKA [8].

§. 1. The unimodular affine group.

We consider the n dimensional affine space and in it the group A of affine transformations modulo 1, which in matrix notation is written

$$x' = Ax + B, \quad \det A = |A| = 1 \tag{1.1}$$

where $A = (a_{ij})$ and $B = (b_i)$ are $n \times n$ and $n \times 1$ matrices respectively; x and x' denote $n \times 1$ matrices whose elements are the n coordinates $x_1, x_2, \ldots, x_n$ of the point x and those $x'_1, x'_2, \ldots, x'_n$ of the transformed point x'.

A transformation of the group A is determined by a pair of matrices (A, B). The identity corresponds to the pair (E, O) where E is the unit matrix and O the $n \times 1$ matrix with all the elements equal to zero. The inverse of (A, B) is

$$(A, B)^{-1} = (A^{-1}, \ -A^{-1}B)$$

and the law for the product can be written

$$(A_2, B_2)(A_1, B_1) = (A_2 A_1, \ A_2 B_1 + B_2).$$

According to the theory of E. CARTAN (see for instance [5]) the *relative components* of the group A will be the elements (which are pfaffian forms) of the matrices Ω_1 (of type $n \times n$) and Ω_2 (of type $n \times 1$) defined by the equation

$$(A, B)^{-1}(A + dA, B + dB) = (E + \Omega_1, \Omega_2).$$

Thus we have

$$\Omega_1 = A^{-1} dA, \quad \Omega_2 = A^{-1} dB. \tag{1.2}$$

By exterior differentiation and taking into account that

$$dA^{-1} = -A^{-1} dA A^{-1} \tag{1.3}$$

we have

$$\begin{aligned} d\Omega_1 &= dA^{-1} \wedge dA = -A^{-1} dA A^{-1} \wedge dA = -\Omega_1 \wedge \Omega_1 \\ d\Omega_2 &= dA^{-1} \wedge dB = -A^{-1} dA A^{-1} \wedge dB = -\Omega_1 \wedge \Omega_2 \end{aligned} \tag{1.4}$$

which are the equations of structure of MAURER-CARTAN for the unimodular affine group A.

In explicit form, if we set

$$A = (a_{ij}), \quad B = (b_i), \quad A^{-1} = (\alpha^{ij}), \Omega_1 = (\omega_{ij}), \Omega_2 = (\omega_i), \tag{1.5}$$

(1.2) and (1.4) can be written in the form

$$\begin{aligned} \omega_{ij} &= \sum_{h=1}^{n} \alpha^{ih} da_{hj} = -\sum_{h=1}^{n} a_{hj} d\alpha^{ih}, \quad \omega_i = \sum_{h=1}^{n} \alpha^{ih} db_h \\ d\omega_{ij} &= -\sum_{h=1}^{n} \omega_{ih} \wedge \omega_{hj}, \quad d\omega_i = -\sum_{h=1}^{n} \omega_{ih} \wedge \omega_h. \end{aligned} \tag{1.6}$$

By differentiation of the condition $\det A = 1$, we get

$$\omega_{11} + \omega_{22} + \cdots + \omega_{nn} = 0. \tag{1.7}$$

§ 2. Density for sets of parallel hyperplanes.

It is known that there does not exist a density for sets of hyperplanes invariant with respect to A [4]. However we are going now to prove that such an invariant density exists for pairs of parallel hyperplanes.

Let us consider two parallel hyperplanes

$$ux = h_1, \quad ux = h_2 \qquad (h_1 \neq h_2) \tag{2.1}$$

where h_1, h_2 are scalars and u a $1 \times n$ matrix $u = (u_1, u_2, \ldots, u_n)$.

By the unimodular affine transformation (A, B) these hyperplanes transform to

$$uA^{-1}(x-B) = h_1, \quad uA^{-1}(x-B) = h_2. \tag{2.2}$$

In order that the varied transformation $(A+dA,\ B+dB)$ may give rise to the same hyperplanes (2.2), the relations

$$d\left(\frac{uA^{-1}}{uA^{-1}B+h_i}\right) = 0 \qquad (i = 1, 2) \tag{2.3}$$

must hold (observe that the denominators are scalars) and we have

$$u\,dA^{-1}(uA^{-1}B+h_i) - uA^{-1}(u\,dA^{-1}B + uA^{-1}dB) = 0.$$

Since dA and dB are independent, a first condition is $uA^{-1}dB = 0$, i.e.

$$u\Omega_2 = 0. \tag{2.4}$$

The remaining terms, give by application of (1.3),

$$u\Omega_1 A^{-1}(h_i + uA^{-1}B) = uA^{-1}(u\Omega_1 A^{-1}B).$$

Since the terms inside the parenthesis are scalars, we can set aside A^{-1} and we have

$$u\Omega_1(h_1 + uA^{-1}B) = u(u\Omega_1 A^{-1}B)$$

$$u\Omega_1(h_2 + uA^{-1}B) = u(u\Omega_1 A^{-1}B)$$

and by substraction

$$u\Omega_1(h_1 - h_2) = 0$$

i. e.

$$u\Omega_1 = 0. \tag{2.5}$$

According to the general theory (see for instance [5]) if we have a set of relative components (or a set of linear combinations of relative components), say $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(n+1)}$ such that the system

$$\omega^{(1)} = 0,\ \omega^{(2)} = 0, \ldots, \omega^{(n+1)} = 0 \tag{2.6}$$

is equivalent to the conditions (2.3), then the density for sets of pairs of parallel hyperplanes (if it exists) will be the exterior product $\omega^{(1)} \wedge \omega^{(2)} \wedge \wedge \cdots \wedge \omega^{(n+1)}$.

In our case the set (2.6) is given by (2.4) (one condition) and (2.5) (n conditions). Since the pairs of parallel hyperplanes are transformed transitively by the group A, without loss of generality we can take

$$(2.7) \qquad u = (0, 0, \ldots, 0, 1), \quad h_1 = 0, \quad h_2 = 1.$$

The transformed hyperplanes (2.2) take then the form

$$(2.8) \qquad \sum_{i=1}^{n} \alpha^{ni} x_i = \sum_{i=1}^{n} \alpha^{ni} b_i, \quad \sum_{i=1}^{n} \alpha^{ni} x_i = \sum_{i=1}^{n} \alpha^{ni} b_i + 1$$

and the system (2.6) will be

$$(2.9) \qquad \omega_n = 0, \quad \omega_{n1} = 0, \quad \omega_{n2} = 0, \ldots, \omega_{nn} = 0.$$

Therefore the density for sets of parallel hyperplanes, when they are written in the form (2.8), reads

$$(2.10) \qquad d\mathfrak{S} = \omega_{n1} \wedge \omega_{n2} \wedge \cdots \wedge \omega_{nn} \wedge \omega_n.$$

The condition for this differential form to be a density is $d(d\mathfrak{S}) = 0$, which is easily verified if we take into account the equations of structure (1.4).

By (1.6) and since $\det A = 1$, we obtain (we always take the densities in absolute value)

$$(2.11) \qquad d\mathfrak{S} = \sum_{i=1}^{n} \alpha^{ni} d\alpha^{n1} \wedge d\alpha^{n2} \wedge \cdots \wedge d\alpha^{nn} \wedge db_i.$$

This formula, together with (2.8) gives the following

Theorem 1. *The density, invariant with respect to the unimodular affine group A, for sets of pairs $\mathfrak{S}$ of parallel hyperplanes written in the form*

$$(2.12) \qquad \sum_{i=1}^{n} l^i x_i = m, \quad \sum_{i=1}^{n} l^i x_i = m + 1$$

is

$$(2.13) \qquad d\mathfrak{S} = dl^1 \wedge dl^2 \wedge \cdots \wedge dl^n \wedge dm.$$

We want now to find a geometrical interpretation of the density [2.13]. Let λ^i be a unit vector. The element of area on the unit hypersphere corresponding to the direction of λ^i is expressed by

$$(2.14) \qquad d\sigma = \frac{d\lambda^2 \wedge d\lambda^3 \wedge \cdots \wedge d\lambda^n}{\lambda^1}$$

and also by

(2.15) $$d\sigma = \sum_{i=1}^{n} (-1)^{i-1} \lambda^i d\lambda^1 \wedge \cdots \wedge d\lambda^{i-1} \wedge d\lambda^{i+1} \wedge \cdots \wedge d\lambda^n.$$

If λ^i is the unit vector normal to the hyperplanes (2.12) and we put

$$\varrho^2 = \sum_{i=1}^{n} (l^i)^2, \qquad d\varrho = \frac{\sum_{i=1}^{n} l^i dl^i}{\varrho}$$

we have

$$\lambda^i = \frac{l^i}{\varrho}, \qquad d\lambda^i = \frac{dl^i}{\varrho} - \frac{l^i}{\varrho^2} d\varrho$$

and

$$d\lambda^2 \wedge d\lambda^3 \wedge \cdots \wedge d\lambda^n = \frac{dl^2 \wedge \cdots \wedge dl^n}{\varrho^{n-1}} - \sum_{i=2}^{n} \frac{l^i dl^2 \wedge \cdots \wedge dl^{i-1} \wedge}{\varrho^n}$$

$$\frac{\wedge d\varrho \wedge dl^{i+1} \wedge \cdots \wedge dl^n}{\varrho^n} = \sum_{i=1}^{n} (-1)^{i-1} \frac{l^1 l^i dl^1 \wedge \cdots \wedge dl^{i-1} \wedge dl^{i+1} \wedge \cdots \wedge dl^n}{\varrho^{n+1}}.$$

Consequently, by (2.14) we obtain

(2.16) $$d\sigma = \frac{1}{\varrho^n} \sum_{i=1}^{n} (-1)^{i-1} l^i dl^1 \wedge \cdots \wedge dl^{i-1} \wedge dl^{i+1} \wedge \cdots \wedge dl^n.$$

Next consider the distances

$$p_1 = \frac{m}{\varrho}, \qquad p_2 = \frac{m+1}{\varrho}$$

from the origin to the hyperplanes (2.12). We have

$$dp_1 = \frac{dm}{\varrho} - \frac{m}{\varrho^2} d\varrho, \qquad dp_2 = \frac{dm}{\varrho} - \frac{(m+1)}{\varrho^2} d\varrho$$

and

$$dp_1 \wedge dp_2 = \frac{1}{\varrho^3} d\varrho \wedge dm = \frac{1}{\varrho^4} \sum_{i=1}^{n} l^i dl^i \wedge dm.$$

Hence

$$d\sigma \wedge dp_1 \wedge ap_2 = \frac{1}{\varrho^{n+2}} dl^1 \wedge dl^2 \wedge \cdots \wedge dl^n \wedge dm = \frac{d\mathfrak{S}}{\varrho^{n+2}}.$$

Remember that we consider always the densities in absolute value; therefore we make no question of te sign.

In order to introduce the distances p_1, p_2 from the origin, we observe that $p_2 - p_1 = 1/\varrho$ and thus

(2.17) $$d\mathfrak{S} = \frac{d\sigma \wedge dp_1 \wedge dp_2}{|p_2 - p_1|^{n+2}}$$

which is the desired geometrical interpretation for $d\mathfrak{S}$.

§. 3. Measure of sets of parallel hyperplanes which contain a given convex body.

Let K be a given convex body in the n dimensional space and let $\Delta=\Delta(\sigma)$ be the width of K corresponding to the direction σ. The measure of all pairs of parallel hyperplanes which contain K will be

$$M=\int\frac{d\sigma\wedge dp_1\wedge dp_2}{|p_2-p_1|^{n+2}}=\frac{1}{n(n+1)}\int\limits_{\frac{1}{2}E}\frac{d\sigma}{\Delta^n}. \tag{3.1}$$

This measures gives, together with its geometrical interpretation, the following affine invariant of K

$$J=\int\limits_{\frac{1}{2}E}\frac{d\sigma}{\Delta^n} \tag{3.2}$$

the integral extended over the half of the n dimensional unit sphere.

Elsewhere [6] we gave an analogous affine invariant I defined by the following integral

$$I=\frac{1}{n}\int\limits_{E}\frac{d\sigma}{p^n} \tag{3.3}$$

where $p=p(\sigma)$ is the support function of K with respect to an interior point also affine invariant with respect to K. In [6] we proved that between I and the volume V of K the inequality

$$IV\leqq\frac{4\pi^n}{n^2(\Gamma(n/2))^2} \tag{3.4}$$

holds, where equality occurs only for ellipsoids.

If K possesses a centre of symmetry, we obviously have $nI=2^{n+1}J$ and the invariant J coincides up to a constant factor with I. If K does not possess a centre of symmetry, J and I are not trivially related. Let us consider the inequalities

$$\frac{1}{x^n}+\frac{1}{y^n}\geqq\frac{2}{(xy)^{n/2}},\quad (xy)^{n/2}\leqq\left(\frac{x+y}{2}\right)^n$$

from which we deduce

$$\frac{1}{x^n}+\frac{1}{y^n}-\frac{2^{n+1}}{(x+y)^n}\geqq 0 \tag{3.5}$$

valid for $x>0$, $y>0$ and where equality occurs only for $x=y$. Denoting by p_1, p_2 the values of p at opposite points and applying (3.5) we have

$$2J=\int_K \frac{d\sigma}{\Delta^n}=\int_K \frac{d\sigma}{(p_1+p_2)^n}\leqq \frac{1}{2^{n+1}}\int_K \left(\frac{d\sigma}{p_1^n}+\frac{d\sigma}{p_2^n}\right)=\frac{n}{2^n}I$$

and therefore

$$J\leqq \frac{n}{2^{n+1}}I \tag{3.6}$$

where equality occurs if and only if K is centrally symmetric. From (3.6) and (3.4) we obtain

$$JV\leqq \frac{4\pi^n}{2^{n+1}n(\Gamma(n/2))^2} \tag{3.7}$$

with equality only for ellipsoids.

In order to obtain a lower bound for the product JV we remind that between the volume P of the least parallelepiped which contains K and the volume V the inequality

$$P\leqq n!\,V$$

holds (see MACBEATH [2]) and that the value of J for a parallelepiped of volume P is

$$J_P=\frac{2^{n-1}}{(n-1)!\,P}$$

as can be obtained by a direct calculation (since J is invariant with respect to affinities it suffices to consider the case of an n-dimensional cube; see, BAMBAH [1]). Hence we have

$$JV\geqq J_P V\geqq \frac{1}{n!}J_P P=\frac{2^{n-1}}{n!\,(n-1)!}\,. \tag{3.8}$$

Since equalities cannot hold simultaneously in (3.8), we always have $JV>2^{n-1}/n!\,(n-1)!$. We may summarize the obtained results as follows

Theorem 2. *The measure of the set of pairs of parallel hyperplanes which contain a given convex body K, invariant with respect to the group of unimodular affine transformations, is given by the integral* (3.1). *This measure gives rise to the invariant J* (3.2) *which is related with the volume V of J by the inequalities*

$$\frac{2^{n-1}}{n!\,(n-1)!}<JV\leqq \frac{4\pi^n}{2^{n+1}n(\Gamma(n/2))^2} \tag{3.9}$$

where the upper bound is attained if and only if K is an ellipsoid.

The exact value of the lower bound is not known. Probably it is attained when K is a simplex, but I have not the proof.

A direct proof of the affine invariance of J together with some generalizations for the cases $n=2,3$ was given elsewhere [7].

§ 4. The real projective group.

Let us now consider the n-dimensional projective space and in it the group of projective transformations

$$x' = Ax, \quad \det A = 1 \tag{4.1}$$

where A is an $(n+1)\times(n+1)$ matrix and x, x' denote $(n+1)\times 1$ matrices whose elements are the homogeneous coordinates $x_0, x_1, \ldots, x_n$ and $x'_0, x'_1, \ldots, x'_n$ of the points x and x' respectively.

Similarly as in the case of the affine group, the relative components ω_{ij} of the projective group are the elements of the matrix

$$\Omega = A^{-1} dA \tag{4.2}$$

and satisfy the equations of structure

$$d\Omega = -\Omega \wedge \Omega. \tag{4.3}$$

If we set

$$A = (a_{ij}), \quad A^{-1} = (\alpha^{ij}), \quad \Omega = (\omega_{ij})$$

the explicit forms of (4.2) and (4.3) are

$$\omega_{ij} = \sum_{h=0}^{n} \alpha^{ih} da_{hj} = -\sum_{h=0}^{n} a_{hj} d\alpha^{ih}, \quad d\omega_{ij} = -\sum_{h=0}^{n} \omega_{ih} \wedge \omega_{hj}. \tag{4.4}$$

By differentiation of the relation $\det A = 1$, we also obtain

$$\omega_{00} + \omega_{11} + \omega_{22} + \cdots + \omega_{nn} = 0. \tag{4.5}$$

5. Measure of sets of hyperquadrics.

Let us consider the hyperquadric

$$x^t \Phi x = 0 \tag{5.1}$$

where x^t denotes the $1\times(n+1)$ transposed matrix of x and Φ is a $(n+1)\times(n+1)$ diagonal matrix

$$\Phi = \begin{pmatrix} \varepsilon_0 & & & \\ & \varepsilon_1 & & \\ & & \ddots & \\ & & & \varepsilon_n \end{pmatrix}$$

with the elements $\varepsilon_i = \pm 1$. It is well known that every hyperquadric is projectively equivalent to one of the type (5.1).

By the projectivity (4.1) the hyperquadric (5.1) transforms to

$$x^t (A^{-1})^t \Phi A^{-1} x = 0. \tag{5.2}$$

In order that the varied projectivity $A + dA$ may conduce to the same hyperquadric, we must have

$$d((A^{-1})^t \Phi A^{-1}) = 0$$

that is, because of (4.2) and (1.3),

$$(A^{-1})^t (\Omega^t \Phi + \Phi \Omega) A^{-1} = 0$$

and thus

$$\Omega^* = \Omega^t \Phi + \Phi \Omega = 0. \tag{5.3}$$

The density for sets of hyperquadrics whose equation has the form (5.2) will be the exterior product of the independent elements of the symmetric matrix Ω^*. These elements are

$$\omega^*_{ij} = \varepsilon_j \omega_{ji} + \varepsilon_i \omega_{ij}$$

and the relation (4.5) gives

$$\sum_{i=0}^{n} \frac{1}{\varepsilon_j} \omega^*_{ii} = 0. \tag{5.4}$$

Therefore the projective invariant density for sets of hyperquadrics may be written in any one of the following equivalent forms (for $i = 0, 1, 2, \dots, n$)

$$dC_i = \omega^*_{00} \wedge \omega^*_{01} \wedge \cdots \wedge \widehat{\omega^*_{ii}} \wedge \cdots \wedge \omega^*_{nn} \tag{5.5}$$

where the hat $\wedge$ means that the covered element must be omitted.

These forms (5.5) are differential forms of degree $\frac{1}{2} n(n+3)$ as it should indeed be. An easy calculation, using the equations of structure (4.4), shows that $d(dC_i) = 0$, and consequently (5.5) is really a projective invariant density for sets of hyperquadrics.

The densities (5.5) refer to the hyperquadric (5.2). It is our purpose now to introduce explicitly the coefficients of this hyperquadric, that is, the elements of the symmetric matrix

$$Q = (A^{-1})^t \Phi A^{-1} \tag{5.6}$$

which are

$$q_{hl} = \sum_{i=0}^{n} \varepsilon_i \alpha^{ih} \alpha^{il}. \tag{5.7}$$

From (5.6) and (5.3) we obtain

(5.8) $$dQ = -(A^{-1})' \Omega^* A^{-1}$$

and hence

(5.9) $$\Omega^* = -A' dQ A$$

or, explicitly

(5.10) $$\omega_{ij}^* = -\sum_{h,l=0}^{n} a_{hi} a_{lj} dq_{hl}.$$

In virtue of the symmetry $q_{ij} = q_{ji}$, the matrix of the system (5.10) (for $i \leqq j$, $h \leqq l$) is the second power matrix of A, denoted by $P_2(A)$ (see, for instance [3 p. 85]) and since $[P_2(A)]^{-1} = P_2(A^{-1})$ (as it follows from the known property $P_2(AB) = P_2(A) P_2(B)$ when we choose $B = A^{-1}$ and remember that $\det A = 1$, $P_2(E) = E$), substituting (5.10) in (5.5) we get

5.11) $$dC_i = \sum_{\substack{h,l=0\\ h\leqq l}}^{n} (-1)^{\nu(i,i)+\nu(h,l)} \alpha^{ih} \alpha^{il} dq_{00} \wedge dq_{01} \wedge \cdots \wedge \widehat{dq}_{hl} \wedge \cdots \wedge dq_{nn}$$

where

(5.12) $$\nu(h,l) = \frac{(2n+1-h)h}{2} + l + 1$$

is the order of the element (h, l) in the sequence $(0,0)$, $(0,1)$, $(0,2), \ldots$, $(h,l), \ldots, (n-1,n), (n,n)$.

Because of (5.4) and (5.5) we observe that $(-1)^{\nu(i,i)} \varepsilon_i dC_i$ does not depend on i; hence we can also take as density for sets of hyperquadrics

(5.13) $$dC = (-1)^{\nu(i,i)} \varepsilon_i dC_i$$

and then, from (5.11) and (5.7) we deduce

(5.14) $$dC = \frac{1}{n+1} \sum_{\substack{h,l=0\\ h\leqq l}}^{n} (-1)^{\nu(h,l)} q_{hl} dq_{00} \wedge dq_{01} \wedge \cdots \wedge \widehat{dq}_{hl} \wedge \cdots \wedge dq_{nn}.$$

This is a first form for dC. A second form is obtained if we observe that by differentiation of $\det Q = \pm 1$. We get

(5.15) $$\sum_{h,l=0}^{n} q^{hl} dq_{hl} = 0, \quad dq_{nn} = -\sum_{h,l=0}^{n}{}' \frac{q^{hl}}{q^{nn}} dq_{hl}$$

where the accent denotes that the term $h = l = n$ is excluded.

Substituting in (5.14) we have, up to the sign which is inessential since we consider always the densities in absolute value,

(5.16) $$dC = \frac{1}{q^{nn}} dq_{00} \wedge dq_{01} \wedge \cdots \wedge dq_{n-1,n}$$

This is a second form for dC.

The densities (5.14) and (5.16) refer to the hyperquadric (5.2) which satisfies the condition $\det Q = \pm 1$. That is, the forms (5.14) and (5.16) apply when we have normalized the equation of the hyperquadric in such a way that $\det Q = \pm 1$ holds, a normalization which is always possible for non degenerate hyperquadrics.

Another normalization could be to take the equation of the hyperquadric in the form

$$\sum_{i,j=0}^{n} q^*_{ij} x_i x_j = 0 \quad \text{with} \quad q^*_{nn} = 1. \tag{5.17}$$

In order to apply to this case the above result, it is enough to set

$$q^*_{ij} = \frac{q_{ij}}{q_{nn}}, \qquad \Delta = \det(q^*_{ij}) = \frac{\det(q_{ij})}{q_{nn}^{n+1}} = \frac{1}{q_{nn}^{n+1}}.$$

We shall have

$$q_{ij} = q^*_{ij} \Delta^{-\frac{1}{n+1}}, \qquad q^{*nn} = \frac{q^{nn}}{q^n_{nn}\Delta} = \frac{q^{nn}}{\Delta^{1/(n+1)}} \tag{5.18}$$

and, since

$$d\Delta = \Delta \sum_{h,l=0}^{n} q^{*hl} dq^*_{hl}$$

we get

$$dq_{ij} = \Delta^{-\frac{1}{n+1}} dq^*_{ij} - \frac{1}{n+1} \Delta^{-\frac{1}{n+1}} q^*_{ij} \sum_{h,l=0}^{n} q^{*hl} dq^*_{hl}$$

and

$$dq_{00} \wedge dq_{01} \wedge \cdots \wedge dq_{n-1,n} = \Delta^{-\frac{n(n+3)}{2(n+1)}} dq^*_{00} \wedge dq^*_{01} \wedge \cdots \wedge dq^*_{n-1,n}$$
$$- \frac{1}{n+1} \Delta^{-\frac{n(n+3)}{2(n+1)}} \sum_{i,j=0}^{n}{}' q^*_{ij} q^{*ij} dq^*_{00} \wedge dq^*_{01} \wedge \cdots \wedge dq^*_{n-1,n}.$$

Since in the last sum the values $i = j = n$ are excluded and $q^*_{nn} = 1$, we have

$$\sum_{i,j=0}^{n}{}' q^*_{ij} q^{*ij} = (n+1) - q^{*nn}$$

and consequently

$$dq_{00} \wedge dq_{01} \wedge \cdots \wedge dq_{n-1,n} = \frac{1}{n+1} \Delta^{-\frac{n(n+3)}{2(n+1)}} q^{*nn} dq^*_{00} \wedge \cdots \wedge dq^*_{n-1,n}.$$

Taking into account (5.16) and (5.18) we finally get, up to the sign,

$$dC = \frac{dq^*_{00} \wedge dq^*_{01} \wedge \cdots \wedge dq^*_{n-1,n}}{(n+1)\Delta^{\frac{n+2}{2}}}. \tag{5.19}$$

For $n=2$, the density for sets of conics written in the form $q_{00}x_0^2+$ $+2q_{01}x_0x_1+2q_{02}x_0x_2+q_{11}x_1^2+2q_{12}x_1x_2+x_2^2=0$ becomes

$$dC=\frac{dq_{00}\wedge dq_{01}\wedge dq_{02}\wedge dq_{11}\wedge dq_{12}}{3\Delta^2}.$$

This expression, up to the factor 1/3 which is inessential, was given by STOKA [8].

We can summarize the above results in the following

Theorem 3. *The projective invariant density for sets of non degenerate hyperquadrics* $x^tQx=0$ *is given by any one of the equivalent forms* (5.14), (5.16) *when det* $Q=\pm 1$. *If the equation of the hyperquadrics satisfies the condition* $q_{nn}=1$ *and we set* $\det Q=\Delta$, *then the density takes the form* (5.19).

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(Received January 27, 1959.)