PORTUGALIAE MATHEMATICA

VOLUME 39

NOTES ON THE INTEGRAL GEOMETRY
IN THE HYPERBOLIC PLANE (*)

BY

L. A. SANTALÓ

Faculdade de Ciências Exactas y Naturales
Universidad de Buenos Aires
Ciudade Universitaria, Buenos Aires
Argentina

ABSTRACT. The Integral Geometry in the hyperbolic plane was initiated, many years ago, in [5]. *Later on, it was applied to the geometry of random mosaics in the hyperbolic plane* [7]. *Im the present work we extend to the hyperbolic plane some new results of the euclidean integral geometry which have been given in recent years for several authors, in particular certain results of H. Hadwiger* [3] *and some formulas of H. J. Firey* [2], *R. Schneider* [8], [9] *and W. Weil* [10] *on the kinematic measure for sets of support figures.*

1. Some elementary remarks and two conjectures

Let K (t) be a family of bounded closed convex sets in the hyperbolic plane, depending on the parameter t (0 ⩽ t) and such that K (t_1) ⊂ K (t_2) for $t_1 < t_2$.

Let F (t) denote the area and L (t) the perimeter of K (t). The isoperimetric inequality

$$(1.1) \qquad\qquad L^2 - 4 \pi F - F^2 \geqslant 0$$

is well known, where the equality sign holds if and only if K is a circle [5], [6, p. 324].

Assume that, for any point P of the plane, there is a value t_p of t such that, for all $t > t_p$, we have $P \in K(t)$. We then say that $K(t)$ expands over the whole plane, as $t \to \infty$.

From (1.1) we deduce

(1.2)
$$\lim_{t \to \infty} \frac{F(t)}{L(t)} \leqslant 1.$$

The convex domain $K(t)$ is said to be h-convex, or convex with respect to horocycles, when for each pair of points A,B belonging to the domain, the entire segments of the two horocycles AB also belong to the domain. Any h-convex set is convex, but the converse is not true. If the boundary $\partial K(t)$ is smooth, the necessary and suffcient condition for h-convexity is that the curvature of ∂K (geodesic curvature) satisfies the condition $\chi_g \geqslant 1$. For instance, the circles are all h-convex. The Gauss-Bonnet formula

(1.3)
$$\int_{\partial k} \chi_g \, ds = 2\pi + F$$

gives then $\lim_{t \to \infty} (F/L) \geqslant 1$ and hence, taking (1.2) into account, we have that for all h-convex sets which expand to the whole hiperbolic plane, the relation

(1.4)
$$\lim_{t \to \infty} \frac{F(t)}{L(t)} = 1$$

holds [7].

We deduce some elementary consequences (the random lines are assumed to be given with the uniform invariant density of the Integral Gometry [6]):

a) If σ denotes the length of the chord that the random line G determines on the h-convex set K, it is well known that the mean value of σ is [6, p. 312]

(1.5)
$$E(\sigma) = \pi F/L.$$

Therefore, according to (1.4), we have: *the mean value of the length σ of the chord that a random line G determines on any h-convex domain $K(t)$, tends to π as $K(t)$ expands to the whole hyperbolic plane.*

b)   Given independently at random two lines G_1, G_2 which inter-sec with a convex set K, the probability that they meet inside K is known to be p $(G_1 \cap G_2 \in K) = 2 \pi F/L^2$. Therefore we have: *the proba-bility that two independent lines wich intersect with a given h-convex set K, meet inside K tends to 0 as K expands to the whole hyperbolic plane.*

c)   The probability that two independent random lines G_1, G_2 which intersect with a given convex domain K be non secant lines in the hyperbolic plane (i.e. $G_1 \cap G_2 = \emptyset$) is

(1.6) $$p\,(G_1 \cap G_2 = \emptyset \mid G_1 \cap K \neq \emptyset, \; G_2 \cap K \neq \emptyset) =$$

$$= 1 - \frac{2 \pi F}{L^2} - \frac{2}{L^2} \int\limits_{P \notin K} (\omega - \sin \omega) \, dP$$

where P is a point exterior to K and ω is the angle between the support lines of K through P. The proof of (1.6) is straightforward by the same method of the euclidean and elliptic planes [6, pp. 51, 319] for which it is p $= 0$.

If K is a circle, by direct computation it is easy to show that the last term of (1.6) tends to 0 as $L \to \infty$. Thus we have: *the probability that two independent random lines which meet a given circle of radius R, be non secant lines of the hyperbolic plane, tends to 1 as $R \to \infty$.*

We end this section with two conjectures:

1)   The equality (1.4) holds good for any convex domain of the hyperbolic plane which expands to the whole plane, i.e. the conditon of h-convexity is superfluous.

2)   The relation lim p $(G_1 \cap G_2 = \emptyset \mid G_1 \cap K \neq \emptyset, \; G_2 \cap K \neq \emptyset) = 1$, as K expands to the whole hyperbolic plane, which we have proven for circles, holds good for any convex set K of the hyperbolic plane.

The proof of these conjectures will be interesting in order to fill several gaps in the integral geometry of the hyperbolic plane.

2.   Generalization of some formulas of Hadwiger to the Hyperbolic plane

2.1.   Let K be a convex set of area F and perimeter L in the hyperbolic plane H^2. The lenght L_ρ and area F_ρ of the set K_ρ parallel to K in the distance ρ, are given by [6, p. 322]

$$
\begin{aligned}
\text{(2.1)} \qquad L_\rho &= (2\pi + F)\ \sinh\rho + L\ \cosh\rho \\
F_\rho &= (2\pi + F)\ \cosh\rho + L\ \sihn\rho - 2\pi.
\end{aligned}
$$

Notice that, if K expands to the whole plane, we have

$$\text{(2.2)} \qquad \lim (L_\rho/L) = \exp\rho \quad,\quad \lim (F_\rho/F) = \exp\rho.$$

In the enclidean plane, both limits are equal to 1.
Let $f(\rho)$ be an integrable function such that

$$\text{(2.3)} \qquad \int_0^\infty f(\rho)\ \sinh\rho\ d\rho < \infty, \quad \int_0^\infty f(\rho)\ \cosh\rho\ d\rho < \infty.$$

A point P exterior to K may be determined by the parameter ρ of the exterior parallel set of K which boundary ∂K_ρ contains P (distance from P to K) and the direction Φ of the tangent line to ∂K_ρ at P. The element of area at P is then $dP = ds_\rho\ d\rho$, where ds_ρ means the arc length of ∂K_ρ at P. Therefore, we have, taking (2.1) into account,

$$\int f(\rho)\ dP = \int_0^\infty f(\rho)\ ds_\rho\ d\rho = \int_0^\infty f(\rho)\ [(2\pi + F)\ \sinh\rho + L\ \cosh\rho]\ d\rho$$

where the first integral is extended to the exterior of K.
This formula may be written

$$\text{(2.4)} \qquad \int_{H_2} f(\rho)\ dP = f(0)\ F + (2\pi + F) \int_0^\infty f(\rho)\ \sinh\rho\ d\rho + $$

$$+ L \int_0^\infty f(\rho)\ \cosh\rho\ d\rho$$

where ρ means the distance from P to K and the integral on the left is extended to the whole hyperbolic plane H^2.
This formula (2.4) generalizes to the hyperbolic plane, an analogous formula given by HADWIGER for the euclidean n-space [3].

EXAMPLE.   For $f(\rho) = \exp(-a\rho)$, $a > 1$, we have

(2.5)
$$\int_0^\infty \exp(-a\rho)\,\sinh\rho\,dP = \frac{1}{a^2 - 1},$$

$$\int_0^\infty \exp(-a\rho)\,\cosh\rho\,d\rho = \frac{a}{a^2 - 1}$$

and therefore we have, for any convex curve K of the hyperbolic plane

(2.6) $$\int_{H^2} \exp(-a\rho)\,dP = \frac{1}{a^2 - 1}(2\pi + aL + a^2F)\quad (a > 1).$$

The analogous formula in the euclidean plane reads

(2.7) $$\int_{E^2} \exp(-a\rho)\,dP = a^{-2}(2\pi + aL + a^2F).$$

2.2.   The density for lines in H^2 is $dG = \cosh\rho\,d\rho\,d\varphi$, where ρ is the distance from G to the origin and φ is the angle of the normal to G through the origin with a fixed direction [6, p. 306]. The measure of the set of lines whose distance to a convex set K lies between ρ and $\rho + d\rho$ is $L_{\rho+d\rho} - L_\rho = L_\rho'\,d\rho$. Therefore, according to (2.1), if $f(\rho)$ is any function with the conditions (2.3), we have

(2.8) $$\int_{H_*^2} f(\rho)\,dG = Lf(0) + \int_0^\infty f(\rho)\,L_\rho'\,d\rho =$$

$$= Lf(0) + (2\pi + F)\int_0^\infty f(\rho)\,\cosh\rho\,d\rho + L\int_0^\infty f(\rho)\,\sinh\rho\,d\rho.$$

For instance, for $f(\rho) = \exp(-a\rho)$, $a > 1$, we have

(2.9) $$\int_{H_*^2} \exp(-a\rho)\,dG = \frac{a}{a^2 - 1}(2\pi + F + aL),\quad a > 1$$

where H_*^2 means the set of all lines of the hyperbolic plane.

The analogous formula for the euclidean plane, reads

$$(2.10) \qquad \int_{E^2_*} \exp(-a\rho) \, dG = a^{-1}(2\pi + aL).$$

2.3.   Let K, K_1 be two convex sets in H^2. We know that the measure of the set of congruent sets to K_1 which intersect with K_ρ (exterior parallel set to K in the distance ρ) is [6, p. 321]

$$(2.11) \quad m(K_\rho \cap K_1 \neq \emptyset) = 2\pi(F_\rho + F_1) + F_\rho F_1 + L_\rho L_1$$

here L_ρ and F_ρ are given by (2.1).

The measure of the sets congruent with K_1 whose distance to K lies between ρ and $\rho + d\rho$, using (2.1) will be

$$(2.12) \quad m_{\rho + d\rho} - m_\rho = m'_\rho d\rho = [(F \sinh \rho + L \cosh \rho + 2\pi \sinh \rho)(2\pi +$$
$$+ F) + (F \cosh \rho + L \sinh \rho + 2\pi \cosh \rho) L] \, d\rho.$$

Therefore, for any function $f(\rho)$ which satisfies the conditions (2.3) we have

$$(2.13) \qquad \int f(\rho) \, dK_1 = f(0) [2\pi(F + F_1) + F F_1 + L L_1]$$
$$+ [2\pi(L + L_1) + L F_1 + F L_1] \int_0^\infty f(\rho) \cosh \rho \, d\rho$$
$$+ [(2\pi + F)(2\pi + F_1) + L L_1] \int_0^\infty f(\rho) \sinh \rho \, d\rho$$

where the integral on the left is extended over all sets congruent to K_1 of the hyperbolic plane.

For euclidean n-spaces, this formula was given by HADWIGER [3]. The classical formula (2.11) corresponds to $f(0) = 1$, $f(\rho) = 0$ for $\rho \neq 0$.

3.   Kinematic measures for sets of support figures

W. J. FIREY [2], R. SCHNEIDER [8], [9] and W. WEIL [10] have considered sets of compact convex sets congruent to K_1 which touch a fixed compact convex set K, i.e. such that K_1 and K have no interior points in common and $\partial K \cap \partial K_1$ is not empty. We want to extend their results to the hyperbolic plane.

From (2.11) and (2.1) we deduce

(3.1)
$$\lim_{\rho \to 0} \rho^{-1} [m (K_\rho \cap K_1 \neq \emptyset) - m (K \cap K_1 \neq \emptyset)] =$$

$$= \left[\frac{d}{d\rho} m (K_\rho \cap K \neq \emptyset) \right]_{\rho = 0} = (2\pi + F_1) L + (2\pi + F) L_1.$$

This expression is called the kinematic measure of positions of K_1 which support K.

If β, β_1, are subsets of the respective boundaries of K and K_1, their normal images on the unit circle Ω can be written, respectively

(3.1)
$$\gamma = \int_\beta K_g \, ds, \qquad \gamma_1 = \int_{\beta_1} K_g^1 \, ds_1$$

where K_g, K_g^1 are the geodesic curvatures and s, s_1 the arc elements on ∂K, ∂K_1.

If l, l_1 are the respective lenghts of the arcs β, β_1, the kinematic measure of positions of K_1 which support K in such a way that β_1 supports β is

(3.2)
$$m (\beta, \beta_1) = \gamma \, l_1 + \gamma_1 \, l.$$

This formula follows from (3.1) noting that the classical formula of Gauss-Bonnet, when $\beta = \partial K$, $\beta_1 = \partial K_1$, gives

$$\gamma = 2\pi + F, \qquad \gamma_1 = 2\pi + F_1.$$

4. Some problems of Geometric Probability

4.1. Consider on the hyperbolic plane H^2 a fixed convex set K. We give at random (uniformly, in the sense of the theory of Geometric Probability) a line-segment S of lenght 1. Assume that S touches K. We want the probability that the contact point $\partial K \cap S$ be an end point of S.

According to (3.1) the total measure of positions of S, is

(4.1)
$$m (\partial K, S) = 2 (2\pi + F) 1 + 2\pi L$$

and the measure of the positions in which S has an end point touching ∂ K corresponds to $l_1 = 0$, i.e. m (∂ K $\cap$ end point of S $\neq \emptyset$) $= 2 \pi$ L. Thus, the probability is

(4.2) $p (\partial K \cap \text{end point of } S \neq \emptyset) = \dfrac{\pi L}{(2 \pi + F) 1 + \pi L}.$

The probability that ∂ K and S touches in an interior point of S will be

(4.3) $p (\partial K \cap \text{interior poit of } S \neq \emptyset) = \dfrac{(2 \pi + F) 1}{(2 \pi + F) 1 + \pi L}.$

If K is h-convex and expands to the whole hyperbolic plane, the probabilities (4.2) and (4.3) tend to $\pi/(\pi + 1)$ and $1/(\pi + 1)$ respectively. In the case of the euclidean plane, these limits are 1 and 0 respectively.

4.2.   Let K be a fixed convex set in H^2 and K_1 a moving triangle $T_1 = ABC$ which touches ∂ K. We want the probability that the contact point be a vertex of the triangle.

According to (3.1) the total measure of contact positions in which the triangle touches K is

$$m_{total} = (2 \pi + F_1) L + (2 \pi + F) L_1$$

where L_1 is the perimeter and F_1 the area of the triangle.

The kinematic measure of the set of positions with a vertex touching ∂ K, according to (3.2) will be $(2 \pi + F_1) L$ and the requested probability results to be

$$p (\partial K \text{ and } T \text{ touches in a vertex}) = \dfrac{(2 \pi + F_1) L}{(2 \pi + F_1) L + (2 \pi + F) L_1}.$$

The probability that ∂ K and T touches in a side of T, is the complementary.

This kind of problems, in euclidean 3-space, were initiated by P. Mc Mullen [4].

5. Line-segment processes in the hyperbolic plane

5.1. The measure of the set of oriented line-segments $S^1 = 0 \, A$ of fixed lenght l_1 which intersect with a given convex set K, according to (2.11) is

$$(5.1) \qquad m \, (K \cap S_1 \neq \varnothing) = 2 \, \pi \, F + 2 \, l_1 \, L$$

and the measure of the oriented line-segment whose origin is interior to K is

$$(5.2) \qquad m \, (S_1 \mid 0 \in K) = 2 \, \pi \, F.$$

The formulas (5.1) and (5.2) are the same for the hyperbolic and for the euclidean plane [6, pp. 90, 321]. Therefore, the probability that a random segment S_1 intersecting with K have its origin inside K, is

$$(5.3) \qquad p \, (0 \in K \mid S_1 \cap K \neq \varnothing) = \frac{\pi \, F}{\pi \, F + l_1 \, L}$$

if K is assumed h-convex and expands to the whole hyperbolic plane (F, L $\to \infty$) this probability tends to 1 for the euclidean plane and to $\pi/(\pi + l_1)$ for the hyperbolic plane. That means that «edge effects» are significant by passing to the limit in the hyperbolic case.

5.2. This «edge effect» is clearly made evident in the following example.

Let K_0 be a convex set contained in K in such a way that any oriented segment of length l_0 which intersects with K_0 has the origin inside K. The probability that a random segment S_1 which has the origin $0 \in K$, intersects with K_0 will be

$$(5.4) \qquad p_1 = \frac{\pi \, F_0 + l \, L_0}{\pi \, F}$$

and the probability that a oriented segment S_1 which intersects with K, also intersects with K_0, will be

$$(5.5) \qquad p_2 = \frac{\pi \, F_0 + l \, L_0}{\pi \, F + l \, L}.$$

Given n random segments with the conditions above, the probabilities that m of them intersect with K_0, will be, respectively (binomial law)

$$(5.6) \qquad p_{1m} = \binom{n}{m} p_1^m (1 - p_1)^{n-m}, \qquad p_{2m} = \binom{n}{m} p_2^m (1 - p_2)^m.$$

Assume that $n \to \infty$ and K expands to the whole hyperbolic plane in such a way that

$$(5.7) \qquad \frac{n}{F} \to \alpha = \text{constant.}$$

The probabilities (5.6) will tend to the limits

$$(5.8) \qquad p_{1m}^* = \frac{(\alpha H_1)^m}{m!} \exp(-\alpha H_1),$$

$$p_{2m}^* = \frac{(\alpha H_2)^m}{m!} \exp(-\alpha H_2)$$

where

$$(5.9) \qquad H_1 = F_0 + \frac{l\, L_0}{\pi}, \qquad H_2 = \frac{\pi F_0 + l\, L_0}{\pi + 1}.$$

Therefore we get two kind of oriented line-segment processes in the hyperbolic plane. The first correspond to a Poisson point process of intensity α, each point being the origin of an oriented segment of length l with the orientation uniformly distributed from 0 to 2π. In this case we have $E(m) = \alpha H_1$, and $E(m) \to \infty$ as $l \to \infty$.

The second process gives

$$E(m) = \alpha \frac{\pi F_0 + l\, L_0}{\pi + 1}$$

and therefore, if $l \to \infty$ we have $E(m) \to \alpha L_0$. We can speak of a process of rays which has no analogous in the euclidean plane.

On line-segment processes in the euclidean plane, see R. COWAN [1].

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