NOTES ON THE INTEGRAL GEOMETRY
IN THE HYPERBOLIC PLANE (*)

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ABSTRACT. The Integral Geometry in the hyperbolic plane was initiated, many years ago, in [5]. Later on, it was applied to the geometry of random mosaics in the hyperbolic plane [7]. In the present work we extend to the hyperbolic plane some new results of the euclidean integral geometry which have been given in recent years for several authors, in particular certain results of H. Hadwiger [3] and some formulas of H. J. Firey [2], R. Schneider [8], [9] and W. Weil [10] on the kinematic measure for sets of support figures.

1. Some elementary remarks and two conjectures

Let \( K(t) \) be a family of bounded closed convex sets in the hyperbolic plane, depending on the parameter \( 0 \leq t \) and such that \( K(t_1) \subseteq K(t_2) \) for \( t_1 < t_2 \).

Let \( F(t) \) denote the area and \( L(t) \) the perimeter of \( K(t) \). The isoperimetric inequality

\[(1.1) \quad L^2 - 4\pi F - F^2 \geq 0\]

is well known, where the equality sign holds if and only if \( K \) is a circle [5], [6, p. 324].

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Assume that, for any point $P$ of the plane, there is a value $t_P$ of $t$ such that, for all $t > t_P$, we have $P \in K(t)$. We then say that $K(t)$ expands over the whole plane, as $t \to \infty$.

From (1.1) we deduce

$$\lim_{t \to \infty} \frac{F(t)}{L(t)} \leq 1.$$  

(1.2)

The convex domain $K(t)$ is said to be h-convex, or convex with respect to horocycles, when for each pair of points $A, B$ belonging to the domain, the entire segments of the two horocycles $AB$ also belong to the domain. Any h-convex set is convex, but the converse is not true. If the boundary $\partial K(t)$ is smooth, the necessary and sufficient condition for h-convexity is that the curvature of $\partial K$ (geodesic curvature) satisfies the condition $\chi_g \geq 1$. For instance, the circles are all h-convex. The Gauss-Bonnet formula

$$\int_{\partial K} \chi_g \, ds = 2\pi + F$$  

(1.3)

gives then $\lim_{t \to \infty} (F/L) \geq 1$ and hence, taking (1.2) into account, we have that for all h-convex sets which expand to the whole hyperbolic plane, the relation

$$\lim_{t \to \infty} \frac{F(t)}{L(t)} = 1$$  

(1.4)

holds [7].

We deduce some elementary consequences (the random lines are assumed to be given with the uniform invariant density of the Integral Geometry [6]):

a) If $\sigma$ denotes the length of the chord that the random line $G$ determines on the h-convex set $K$, it is well known that the mean value of $\sigma$ is [6, p. 312]

$$\mathbb{E}(\sigma) = \pi \frac{F}{L}.$$  

(1.5)

Therefore, according to (1.4), we have: the mean value of the length $\sigma$ of the chord that a random line $G$ determines on any h-convex domain $K(t)$, tends to $\pi$ as $K(t)$ expands to the whole hyperbolic plane.
b) Given independently at random two lines \( G_1, G_2 \) which intersect with a convex set \( K \), the probability that they meet inside \( K \) is known to be \( p(G_1 \cap G_2 \in K) = 2\pi F/L^2 \). Therefore we have: the probability that two independent lines which intersect with a given \( h \)-convex set \( K \), meet inside \( K \) tends to 0 as \( K \) expands to the whole hyperbolic plane.

c) The probability that two independent random lines \( G_1, G_2 \) which intersect with a given convex domain \( K \) be non secant lines in the hyperbolic plane (i.e. \( G_1 \cap G_2 = \emptyset \)) is

\[
(1.6) \quad p(G_1 \cap G_2 = \emptyset | G_1 \cap K \neq \emptyset, \ G_2 \cap K \neq \emptyset) = 1 - \frac{2\pi F}{L^2} - \frac{2}{L^2} \int_{P \in K} (\omega - \sin \omega) \, dP
\]

where \( P \) is a point exterior to \( K \) and \( \omega \) is the angle between the support lines of \( K \) through \( P \). The proof of (1.6) is straightforward by the same method of the euclidean and elliptic planes \([6, \text{pp. 51, 319}]\) for which it is \( p = 0 \).

If \( K \) is a circle, by direct computation it is easy to show that the last term of (1.6) tends to 0 as \( L \to \infty \). Thus we have: the probability that two independent random lines which meet a given circle of radius \( R \), be non secant lines of the hyperbolic plane, tends to 1 as \( R \to \infty \).

We end this section with two conjectures:

1) The equality (1.4) holds good for any convex domain of the hyperbolic plane which expands to the whole plane, i.e. the condition of \( h \)-convexity is superfluous.

2) The relation \( \lim p(G_1 \cap G_2 = \emptyset | G_1 \cap K \neq \emptyset, \ G_2 \cap K \neq \emptyset) = 1 \), as \( K \) expands to the whole hyperbolic plane, which we have proven for circles, holds good for any convex set \( K \) of the hyperbolic plane.

The proof of these conjectures will be interesting in order to fill several gaps in the integral geometry of the hyperbolic plane.
2. Generalization of some formulas of Hadwiger to the Hyperbolic plane

2.1. Let $K$ be a convex set of area $F$ and perimeter $L$ in the hyperbolic plane $H^2$. The length $L_p$ and area $F_p$ of the set $K_p$ parallel to $K$ in the distance $p$, are given by [6, p. 322]

\[ L_p = (2\pi + F) \sinh p + L \cosh p \]

\[ F_p = (2\pi + F) \cosh p + L \sinh p - 2\pi. \]

Notice that, if $K$ expands to the whole plane, we have

\[ \lim (L_p/L) = \exp p, \quad \lim (F_p/F) = \exp p. \]

In the euclidean plane, both limits are equal to 1.

Let $f(p)$ be an integrable function such that

\[ \int f(p) \sinh p \; dp < \infty, \quad \int f(p) \cosh p \; dp < \infty. \]

A point $P$ exterior to $K$ may be determined by the parameter $p$ of the exterior parallel set of $K$ which boundary $\partial K_p$ contains $P$ (distance from $P$ to $K$) and the direction $\Phi$ of the tangent line to $\partial K_p$ at $P$. The element of area at $P$ is then $dP = ds_p \; dp$, where $ds_p$ means the arc length of $\partial K_p$ at $P$. Therefore, we have, taking (2.1) into account,

\[ \int f(p) \; dP = \int f(p) \; ds_p \; dp = \int f(p) [(2\pi + F) \sinh p + L \cosh p] \; dp \]

where the first integral is extended to the exterior of $K$.

This formula may be written

\[ \int f(p) \; dP = f(0) F + (2\pi + F) \int \tilde{f}(p) \; \sinh p \; dp + L \int \tilde{f}(p) \; \cosh p \; dp \]

where $p$ means the distance from $P$ to $K$ and the integral on the left is extended to the whole hyperbolic plane $H^2$.

This formula (2.4) generalizes to the hyperbolic plane, an analogous formula given by Hadwiger for the euclidean $n$-space [3].
Example. For \( f(p) = \exp(-ap) \), \( a > 1 \), we have

\[
\int_0^\infty \exp(-ap) \sinh \rho \, d\rho = \frac{1}{a^2 - 1},
\]

\[
\int_0^\infty \exp(-ap) \cosh \rho \, d\rho = \frac{a}{a^2 - 1}
\]

and therefore we have, for any convex curve \( K \) of the hyperbolic plane

\[
\int_{H^2} \exp(-ap) \, d\rho = \frac{1}{a^2 - 1} (2\pi + aL + a^2F) \quad (a > 1).
\]

The analogous formula in the Euclidean plane reads

\[
\int_{E^2} \exp(-ap) \, d\rho = a^{-2} (2\pi + aL + a^2F).
\]

2.2. The density for lines in \( H^2 \) is \( dG = \cosh \rho \, d\rho \, d\varphi \), where \( \rho \) is the distance from \( G \) to the origin and \( \varphi \) is the angle of the normal to \( G \) through the origin with a fixed direction [6, p. 306]. The measure of the set of lines whose distance to a convex set \( K \) lies between \( \rho \) and \( \rho + d\rho \) is \( L_{\rho + d\rho} - L_{\rho} = L_{\rho} \, d\rho \). Therefore, according to (2.1), if \( f(\rho) \) is any function with the conditions (2.3), we have

\[
\int_{H^2} f(\varphi) \, dG = L f(0) + \int_0^\infty f(\varphi) \, L_{\rho} \, d\rho = L f(0) + (2\pi + F) \int_0^\infty f(\varphi) \cosh \varphi \, d\varphi + L \int_0^\infty f(\varphi) \sinh \varphi \, d\varphi.
\]

For instance, for \( f(\rho) = \exp(-ap) \), \( a > 1 \), we have

\[
\int_{H^2} \exp(-ap) \, dG = \frac{a}{a^2 - 1} (2\pi + F + aL), \quad a > 1
\]

where \( H^2 \) means the set of all lines of the hyperbolic plane.
The analogous formula for the euclidean plane, reads

\[
\int_{E^2} \exp(-a\varphi) \, dG = a^{-1} (2\pi + aL).
\]

2.3. Let \( K, K_1 \) be two convex sets in \( H^2 \). We know that the measure of the set of congruent sets to \( K_1 \) which intersect with \( K_\rho \) (exterior parallel set to \( K \) in the distance \( \rho \)) is [6, p. 321]

\[
m(K_\rho \cap K_1 \neq \emptyset) = 2\pi (F_\rho + F_1) + F_\rho F_1 + L_\rho L_1
\]

here \( L_\rho \) and \( F_\rho \) are given by (2.1).

The measure of the sets congruent with \( K_1 \) whose distance to \( K \) lies between \( \rho \) and \( \rho + d\rho \), using (2.1) will be

\[
m_\rho + d\rho - m_\rho = m_\rho' d\rho = [(F \sinh \rho + L \cosh \rho + 2\pi \sinh \rho) (2\pi + F) + (F \cosh \rho + L \sinh \rho + 2\pi \cosh \rho) L] \, d\rho.
\]

Therefore, for any function \( f(\rho) \) which satisfies the conditions (2.3) we have

\[
\int f(\rho) \, dK_1 = f(0) [2\pi (F + F_1) + F F_1 + L L_1]
+ [2\pi (L + L_1) + L F_1 + F L_1] \int_0^\infty f(\rho) \cosh \rho \, d\rho
+ [(2\pi + F) (2\pi + F_1) + L L_1] \int_0^\infty f(\rho) \sinh \rho \, d\rho
\]

where the integral on the left is extended over all sets congruent to \( K_1 \) of the hyperbolic plane.

For euclidean \( n \)-spaces, this formula was given by Hadwiger [3]. The classical formula (2.11) corresponds to \( f(0) = 1, f(\rho) = 0 \) for \( \rho \neq 0 \).

3. Kinematic measures for sets of support figures

W. J. Firey [2], R. Schneider [8], [9] and W. Weil [10] have considered sets of compact convex sets congruent to \( K_1 \) which touch a fixed compact convex set \( K \), i.e. such that \( K_1 \) and \( K \) have no interior points in common and \( \partial K \cap \partial K_1 \) is not empty. We want to extend their results to the hyperbolic plane.
From (2.11) and (2.1) we deduce

\[
\lim_{\rho \to 0} \rho^{-1} \left[ m(K_\rho \cap K_1 \neq \emptyset) - m(K \cap K_1 \neq \emptyset) \right] =
\]

\[
= \left[ \frac{d}{d\rho} m(K_\rho \cap K \neq \emptyset) \right]_{\rho \to 0} = (2\pi + F_1) L + (2\pi + F) L_1.
\]

This expression is called the kinematic measure of positions of \(K_1\) which support \(K\).

If \(\beta, \beta_1\) are subsets of the respective boundaries of \(K\) and \(K_1\), their normal images on the unit circle \(\Omega\) can be written, respectively

\[
\gamma = \int_{\beta} K_s \, ds, \quad \gamma_1 = \int_{\beta_1} K'_s \, ds_1
\]

where \(K_s, K'_s\) are the geodesic curvatures and \(s, s_1\) the arc elements on \(\partial K, \partial K_1\).

If \(l, l_1\) are the respective lengths of the arcs \(\beta, \beta_1\), the kinematic measure of positions of \(K_1\) which support \(K\) in such a way that \(\beta_1\) supports \(\beta\) is

\[
m(\beta, \beta_1) = \gamma l_1 + \gamma_1 l.
\]

This formula follows from (3.1) noting that the classical formula of Gauss-Bonnet, when \(\beta = \partial K, \beta_1 = \partial K_1\), gives

\[
\gamma = 2\pi + F, \quad \gamma_1 = 2\pi + F_1.
\]

4. Some problems of Geometric Probability

4.1. Consider on the hyperbolic plane \(H^2\) a fixed convex set \(K\). We give at random (uniformly, in the sense of the theory of Geometric Probability) a line-segment \(S\) of length \(l\). Assume that \(S\) touches \(K\). We want the probability that the contact point \(\partial K \cap S\) be an end point of \(S\).

According to (3.1) the total measure of positions of \(S\), is

\[
m(\partial K, S) = 2(2\pi + F) l + 2\pi L
\]
and the measure of the positions in which $S$ has an end point touching $\partial K$ corresponds to $l_1 = 0$, i.e. $m(\partial K \cap \text{end point of } S \neq \emptyset) = 2\pi L$. Thus, the probability is

$$p(\partial K \cap \text{end point of } S \neq \emptyset) = \frac{\pi L}{(2\pi + F)1 + \pi L}.$$  

The probability that $\partial K$ and $S$ touches in an interior point of $S$ will be

$$p(\partial K \cap \text{interior point of } S \neq \emptyset) = \frac{(2\pi + F)1}{(2\pi + F)1 + \pi L}.$$  

If $K$ is $h$-convex and expands to the whole hyperbolic plane, the probabilities (4.2) and (4.3) tend to $\pi/(\pi + 1)$ and $1/(\pi + 1)$ respectively. In the case of the euclidean plane, these limits are 1 and 0 respectively.

4.2. Let $K$ be a fixed convex set in $H^2$ and $K_1$ a moving triangle $T_1 = ABC$ which touches $\partial K$. We want the probability that the contact point be a vertex of the triangle.

According to (3.1) the total measure of contact positions in which the triangle touches $K$ is

$$m_{\text{total}} = (2\pi + F_1) L = (2\pi + F) L_1$$

where $L_1$ is the perimeter and $F_1$ the area of the triangle.

The kinematic measure of the set of positions with a vertex touching $\partial K$, according to (3.2) will be $(2\pi + F_1) L$ and the requested probability results to be

$$p(\partial K \text{ and } T \text{ touches in a vertex}) = \frac{(2\pi + F_1) L}{(2\pi + F_1) L + (2\pi + F) L_1}.$$  

The probability that $\partial K$ and $T$ touches in a side of $T$, is the complementary.

This kind of problems, in euclidean 3-space, were initiated by P. Mc Mullen (4).
5. Line-segment processes in the hyperbolic plane

5.1. The measure of the set of oriented line-segments $S = OA$ of fixed length $l_1$ which intersect with a given convex set $K$, according to (2.11) is

$$m(K \cap S \neq \emptyset) = 2\pi F + 2l_1L$$

and the measure of the oriented line-segment whose origin is interior to $K$ is

$$m(S_1 \mid 0 \in K) = 2\pi F.$$

The formulas (5.1) and (5.2) are the same for the hyperbolic and for the euclidean plane [6, pp. 90, 321]. Therefore, the probability that a random segment $S_1$ intersecting with $K$ have its origin inside $K$, is

$$p(0 \in K \mid S_1 \cap K \neq \emptyset) = \frac{\pi F}{\pi F + l_1L}$$

if $K$ is assumed h-convex and expands to the whole hyperbolic plane $(F, L \to \infty)$ this probability tends to 1 for the euclidean plane and to $\pi/(\pi + l_1)$ for the hyperbolic plane. That means that «edge effects» are significant by passing to the limit in the hyperbolic case.

5.2. This «edge effect» is clearly made evident in the following example.

Let $K_0$ be a convex set contained in $K$ in such a way that any oriented segment of length $l_0$ which intersects with $K_0$ has the origin inside $K$. The probability that a random segment $S_1$, which has the origin $0 \in K$, intersects with $K_0$ will be

$$p_1 = \frac{\pi F_0 + l_0}{\pi F}$$

and the probability that a oriented segment $S_1$ which intersects with $K$, also intersects with $K_0$, will be

$$p_2 = \frac{\pi F_0 + l_0}{\pi F + l}.$$
Given \( n \) random segments with the conditions above, the probabilities that \( m \) of them intersect with \( K_n \) will be, respectively (binomial law)

\[
\begin{align*}
\rho_{1m} &= \binom{n}{m} p_1^m (1 - p_1)^{n-m}, \\
\rho_{2m} &= \binom{n}{m} p_2^m (1 - p_2)^m.
\end{align*}
\]

Assume that \( n \to \infty \) and \( K \) expands to the whole hyperbolic plane in such a way that

\[
\frac{n}{F} \to \alpha = \text{constant}.
\]

The probabilities (5.6) will tend to the limits

\[
\begin{align*}
\rho_{1m}^* &= \frac{(\alpha H_1)^m}{m!} \exp(-\alpha H_1), \\
\rho_{2m}^* &= \frac{(\alpha H_2)^m}{m!} \exp(-\alpha H_2)
\end{align*}
\]

where

\[
H_1 = F_0 + \frac{1 L_0}{\pi}, \quad H_2 = \frac{\pi F_0 + 1 L_0}{\pi + 1}.
\]

Therefore we get two kind of oriented line-segment processes in the hyperbolic plane. The first correspond to a Poisson point process of intensity \( \alpha \), each point being the origin of an oriented segment of length 1 with the orientation uniformly distributed from 0 to \( 2\pi \). In this case we have \( E(m) = \alpha H_1 \), and \( E(m) \to \infty \) as \( l \to \infty \).

The second process gives

\[
E(m) = \frac{\pi F_0 + 1 L_0}{\pi + 1}
\]

and therefore, if \( l \to \infty \) we have \( E(m) \to \alpha L_0 \). We can speak of a process of rays which has no analogous in the euclidean plane.

On line-segment processes in the euclidean plane, see R. Cowan [1].
REFERENCES