SPACES WITH TWO AFFINE CONNECTIONS

By

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Dedicated to the memory of late Prof. Syamadas Mukhopadhyay on the occasion of his birth centenary

1. Introduction. In an n-dimensional space with several affine connections ${}^{\alpha}\Gamma_{ij}^{s}$ $(\alpha = 1, 2, 3, \ldots)$ it is possible to define certain tensors analogous to the ordinary curvature tensor. They may be obtained as coefficients of the generalized Ricci identities of the classical Riemannian geometry, as was done by the present author in a previous paper (Stantaló, 1954). Recently, following a similar way, Sen (Sen, 1964) has obtained new tensors and new quantities, which he applies to get generalized forms of the equations of the unified field theory of Einstein, as developed by Hlavatý (Hlavatý, 1957).

In the present paper we consider especially the case of two connections ${}^{a}\Gamma$, ${}^{b}\Gamma$ and obtain several tensors (among them those of Sen) and certain properties of these tensors.

2. Generalized Covariant Differentiation. Let A_i be a covariant vector in an n-dimensional space in which several affine connections ${}^{\alpha}\Gamma_{ij}^{s}$ $(\alpha = 1, 2, 3, \ldots)$ are given. We shall set throughout the paper

$$ {}^{\alpha'}\Gamma_{ij}^{s} = {}^{\alpha}\Gamma_{ji}^{s}. \tag{2.1} $$

The covariant derivative of A_i with respect to ${}^{1}\Gamma$ is

$$ A_{i\underset{1}{;}j} = A_{i,j} - {}^{1}\Gamma_{i}{}^{s}A_{s} \tag{2.2} $$

where a semi-colon indicates covariant differentiation and a comma ordinary partial differentiation.

A further covariant differentiation, first with respect to ${}^{2}\Gamma$ and then with respect to ${}^{3}\Gamma$ yields

$$ A_{i\underset{\substack{1\\2}}{;}\underset{3}{j}h} = (A_{i,jh} - {}^{1}\Gamma_{ij,h}^{s}A_{s} - {}^{1}\Gamma_{ij}^{s}A_{s,h}) - {}^{2}\Gamma_{ih}^{m}(A_{m,j} - {}^{1}\Gamma_{mj}^{s}A_{s}) $$
$$ - {}^{3}\Gamma_{jh}^{m}(A_{i,m} - {}^{1}\Gamma_{im}^{s}A_{s}). \tag{2.3} $$

Analogously we have

$$ A_{i\underset{\substack{4\\5}}{;}\underset{6}{h}j} = (A_{i,hj} - {}^{4}\Gamma_{ih,j}^{s}A_{s} - {}^{4}\Gamma^{s}{}_{h}A_{s,j}) - {}^{5}\Gamma_{ij}^{m}(A_{m,h} - {}^{4}\Gamma_{mh}^{s}A_{s}) $$
$$ - {}^{6}\Gamma_{hj}^{m}(A_{i,m} - {}^{4}\Gamma_{im}^{s}A_{s}). \tag{2.4} $$

From (2.3) and (2.4) we deduce

$$ A_{i\underset{\substack{1\\2}}{;}\underset{3}{j}h} - A_{i\underset{\substack{4\\5}}{;}\underset{6}{h}j} = A_{s}({}^{4}\Gamma_{ih,j}^{s} - {}^{1}\Gamma_{ij,h}^{s} + {}^{1}\Gamma^{s}{}_{mj}{}^{2}\Gamma_{ih}^{m} - {}^{4}\Gamma^{s}{}_{mh}{}^{5}\Gamma_{ij}^{m} $$
$$ + {}^{1}\Gamma_{im}^{s}{}^{3}\Gamma_{jh}^{m} - {}^{4}\Gamma_{im}^{s}{}^{6}\Gamma_{hj}^{m}) + A_{i,m}({}^{6}\Gamma_{hj}^{m} - {}^{3}\Gamma_{jh}^{m}) + A_{m,j}({}^{4}\Gamma_{ih}^{m} - {}^{2}\Gamma_{ih}^{m}) $$
$$ + A_{m,h}({}^{5}\Gamma_{ij}^{m} - {}^{1}\Gamma_{ij}^{m}). \tag{2.5} $$

In order to set the right member as a sum of tensors, we put

$$A_{i,m} = A_{\underset{7}{i};m}+{}^7\Gamma^s_{im}A_s,\quad A_{m,j} = A_{\underset{8}{m};j}+{}^8\Gamma^s_{mj}A_s,\quad A_{m,h} = A_{\underset{9}{m};h}+{}^9\Gamma^s_{mh}A_s \tag{2.6}$$

and we get the following generalized identities of Ricci

$$A_{\underset{1\,2}{i};\underset{3}{jh}}-A_{\underset{4\,5}{i};\underset{6}{hj}} = A_sQ^s_{ijh}(1, 2, \dots, 9)+A_{\underset{7}{i};m}({}^6\Gamma^m_{hj}-{}^3\Gamma^m_{jh})$$

$$+A_{\underset{8}{m};j}({}^4\Gamma^m_{ih}-{}^2\Gamma^m_{ih})+A_{\underset{9}{m};h}({}^5\Gamma^m_{ij}-{}^1\Gamma^m_{ij}) \tag{2.7}$$

where

$$Q^s_{ijh}(1, 2, \dots, 9) = {}^4\Gamma^s_{ih,j}-{}^1\Gamma^s_{ij,h}+{}^1\Gamma^s_{mj}{}^2\Gamma^m_{ih}-{}^4\Gamma^s_{mh}{}^5\Gamma^m_{ij}+{}^1\Gamma^s_{im}{}^3\Gamma^m_{jh}$$

$$-{}^4\Gamma^s_{im}{}^6\Gamma^m_{hj}+{}^7\Gamma^s_{im}({}^6\Gamma^m_{hj}-{}^3\Gamma^m_{jh})+{}^8\Gamma^s_{mj}({}^4\Gamma^m_{ih}-{}^2\Gamma^m_{ih})+{}^9\Gamma^s_{mh}({}^5\Gamma^m_{ij}-{}^1\Gamma^m_{ij}). \tag{2.8}$$

From (2.7) it is clear that $Q^s_{ijh}(1, 2, \dots, 9)$ is a tensor for any ${}^\alpha\Gamma(\alpha = 1, 2, \dots, 9)$.

Proceeding in like manner with a contravariant vector A^i we get the same formula (2.7) with the tensor

$${}^*Q^s_{ijh}(1, 2, \dots, 9) = {}^4\Gamma^s_{ih,j}-{}^1\Gamma^s_{ij,h}-{}^2\Gamma^s_{mh}{}^1\Gamma^m_{ij}+{}^1\Gamma^s_{im}{}^3\Gamma^m_{jh}+{}^5\Gamma^s_{mj}{}^4\Gamma^m_{ih}$$

$$-{}^4\Gamma^s_{im}{}^6\Gamma^m_{hj}+({}^1\Gamma^s_{mj}-{}^5\Gamma^s_{mj}){}^9\Gamma^m_{ih}+({}^2\Gamma^s_{mh}-{}^4\Gamma^s_{mh})\,{}^8\Gamma^m_{ij}+{}^7\Gamma^s_{im}({}^6\Gamma^m_{hj}-{}^3\Gamma^m_{jh}) \tag{2.9}$$

instead of (2.8).

Hence we have the following theorem

Theorem. *In order that the generalized covariant differentiation of all vectors with respect to the connections* ${}^\alpha\Gamma$ $(\alpha = 1, 2, \dots, 6)$ *taken in the order indicated in the left side of* (2.7) *be commutative, it is necessary and sufficient that*

$${}^4\Gamma = {}^2\Gamma,\quad {}^5\Gamma = {}^1\Gamma,\quad {}^6\Gamma = {}^{3'}\Gamma \tag{2.10}$$

and that

$$R^s_{ijh}(1, 2, 3) = Q^s_{ijh}(1, 2, 3, 2, 1, 3', 7, 8, 9) = 0, \tag{2.11}$$

where the tensors $R^s_{ijh}(1, 2, 3)$ *have the form*

$$R^s_{ijh}(1, 2, 3) = {}^2\Gamma^s_{ih,j}-{}^1\Gamma^s_{ij,h}+{}^1\Gamma^s_{mj}{}^2\Gamma^m_{ih}-{}^2\Gamma^s_{mh}{}^1\Gamma^m_{ij}+({}^1\Gamma^s_{im}-{}^2\Gamma^s_{im}){}^3\Gamma^m_{jh}. \tag{2.12}$$

3. The Case of two Connections. Let us consider the case of a space with only two connections ${}^a\Gamma$, ${}^b\Gamma$. We wish to write down all possible tensors (2.8) where the indices 1, 2, ..., 9 must be either a or b. In order to do that we introduce the following tensors

$${}^aS^s_{ij} = {}^a\Gamma^s_{ij}-{}^a\Gamma^s_{ji},\quad {}^bS^s_{ij} = {}^b\Gamma^s_{ij}-{}^b\Gamma^s_{ji},\quad T^s_{ij} = T^s_{ij}(a, b) = {}^a\Gamma^s_{ij}-{}^b\Gamma^s_{ij} \tag{3.1}$$

$$R^s_{ijh}(a, b; c) = {}^a\Gamma^s_{ih,j}-{}^b\Gamma^s_{ij,h}+{}^b\Gamma^s_{mj}{}^a\Gamma^m_{ih}-{}^a\Gamma^s_{mh}{}^b\Gamma^m_{ij}-T^s_{im}{}^c\Gamma^m_{jh} \tag{3.2}$$

where c can have the values a, b, a', b' (2.1).

Notice that

$$R^s_{ijh}(a, a; c) = R^s_{ijh}(a),\quad R^s_{ijh}(b, b;\ c) = R^s_{ijh}(b)$$

are the ordinary curvature tensors with respect to the connections ${}^a\Gamma$, ${}^b\Gamma$ respectively.

Then, a rather long but straightforward computation shows that *all the tensors* Q^{s}_{ijh} (1, 2, ..., 9) *reduce to linear combinations of the following tensors*

$$T^{s}_{im}{}^{c}S^{m}_{hj},\ T^{s}_{im}T^{m}_{hj},\ R^{s}_{ijh}(a, b; c)\ (c = a, b, a', b') \tag{3.3}$$

included the isomers of $T^{s}_{im}T^{m}_{hj}$ (i.e.. *those tensors obtained by changing the location of the indices*). Among the tensors $R^{s}_{ijh}(a, b; c)$ we include $R^{s}_{ijh}(a, a; c)$, $R^{s}_{ijh}(b, b; c)$ and $R^{s}_{ijh}(b, a; c)$.

Therefore in the spaces with two affine connections ${}^{a}\Gamma$, ${}^{b}\Gamma$ besides the ordinary curvature tensors $R^{s}_{ijh}(a)$, $R^{s}_{ijh}(b)$ play a fundamental role the tensors $R^{s}_{ijh}(a, b; c)$, given by (3.2).

If we wish to eliminate the connection ${}^{c}\Gamma$, we take the sum $R^{s}_{ijh}(a, b; c)+R^{s}_{ijh}(b, a; c)$ and having into account that $T^{s}_{ij}(a, b) = -T^{s}_{ij}(b, a)$, we get the tensor

$$\begin{aligned}{}^{1}R^{s}{}_{ijh} &= R^{s}{}_{ijh}(a, b; c)+R^{s}{}_{ijh}(b, a; c) = {}^{a}\Gamma^{s}{}_{ih,j}+{}^{b}\Gamma^{s}{}_{ih,j}\\ &-{}^{a}\Gamma^{s}_{ij,h}-{}^{b}\Gamma^{s}_{ij,h}+{}^{a}\Gamma^{s}_{mj}{}^{b}\Gamma^{m}_{ih}+{}^{b}\Gamma^{s}_{mj}{}^{a}\Gamma^{m}_{ih}-{}^{a}\Gamma^{s}_{mh}{}^{b}\Gamma^{m}_{ij}-{}^{b}\Gamma^{s}_{mh}{}^{a}\Gamma^{m}_{ij},\end{aligned} \tag{3.4}$$

which, up to a factor $\frac{1}{2}$, coincides with the first tensor of Sen (Sen, 1964). The second tensor of Sen is

$${}^{2}R^{s}_{ijh} = {}^{1}R^{s}_{ijh}-T^{s}_{mh}T^{m}_{ij} \tag{3.5}$$

The tensors $R^{s}_{ijh}(a, b; c)$ satisfy the following identities

$$R^{s}_{ijh}(a, b; c)-R^{s}_{ihj}(a, b; c_1) = {}^{1}R^{s}_{ijh}+T^{s}_{im}({}^{c_1}\Gamma^{m}_{hj}-{}^{c}\Gamma^{m}_{jh}) \tag{3.6}$$

$$R^{s}_{ijh}(a, b; c)+R^{s}_{ihj}(b, a; c_1) = T^{s}_{im}({}^{c_1}\Gamma^{m}_{hj}-{}^{c}\Gamma^{m}_{jh}). \tag{3.7}$$

In particular, we have

$$R^{s}_{ijh}(a, b; c)-R^{s}_{ihj}(a, b; c') = {}^{1}R^{s}_{ijh} \tag{3.8}$$

$$R^{s}_{ijh}(a, b; c)+R^{s}_{ihj}(b, a; c') = 0. \tag{3.9}$$

From (3.9) and (3.4) follows the relation of Sen (Sen, 1964)

$${}^{1}R^{s}_{ijh}+{}^{1}R^{s}_{ihj} = 0.$$

4. Contracted Tensors.

From (3.2) we deduce the following contracted tensors

$$\begin{aligned}R_{ij}(a, b; c) &= R^{s}_{ijs}(a, b; c) = {}^{a}\Gamma^{s}_{is,j}-{}^{b}\Gamma^{s}_{ij,s}+{}^{b}\Gamma^{s}_{mj}{}^{a}\Gamma^{m}_{is}\\ &-{}^{a}\Gamma^{s}_{ms}{}^{b}\Gamma^{m}_{ij}-T^{s}_{im}{}^{c}\Gamma^{m}_{js}.\end{aligned} \tag{4.1}$$

$$R^{*}{}_{ij}(a, b; c) = R^{s}_{isj}(a, b; c) = {}^{a}\Gamma^{s}_{ij,s}-{}^{b}\Gamma^{s}_{is,j}+{}^{b}\Gamma^{s}_{ms}{}^{a}\Gamma^{m}_{ij}-{}^{a}\Gamma^{s}_{mj}{}^{b}\Gamma^{m}_{is}-T^{s}_{im}\Gamma^{m}_{sj}. \tag{4.2}$$

$$R^{**}{}_{ij}(a, b; c) = R^{s}_{sij}(a, b; c) = {}^{a}\Gamma^{s}_{sj,i}-{}^{b}\Gamma^{s}_{si,j}-{}^{b}T^{s}_{sm}{}^{c}\Gamma^{m}_{ij}. \tag{4.3}$$

which satisfy the following identities

$$R^{*}{}_{ij}(a, b; c)+R_{ij}(b, a; c') = 0 \tag{4.4}$$

$$R^{**}{}_{ij}(a, b; c)+R^{**}{}_{ij}(b, a; c') = 0. \tag{4.5}$$

5. The two connections of a Group of Lie. A well known example of spaces with two connections is the space of the simply transitive Lie groups; see, for instance (Eisenhart, 1933, p. 192-198) and (Schouten, 1954, p. 185-191). The two connections are in this case alternate to each other, that is

$$^{b}\Gamma^{i}_{ij} = {}^{a'}\Gamma^{i}_{ij} = {}^{a}\Gamma^{i}_{ji}. \tag{5.1}$$

If we denote by Γ^{i}_{ij} the symmetric part of $^{a}\Gamma^{i}_{ij}$ and by Ω^{i}_{ij} its tensor of torsion, then

$$^{a}\Gamma^{i}_{ij} = \Gamma^{i}_{ij}+\Omega^{i}_{ij}, \quad {}^{b}\Gamma^{i}_{ij} = \Gamma^{i}_{ij}-\Omega^{i}_{ij}. \tag{5.2}$$

The tensor (3.2) becomes

$$R^{i}_{ijh}(a, a'; a) = B^{i}_{ijh}+\Omega^{i}_{ih;j}+\Omega^{i}_{ij;h}+\Omega^{i}_{mh}\Omega^{m}_{ij}-\Omega^{i}_{mj}\Omega^{m}_{ih}-2\Omega^{i}_{im}\Omega^{m}_{jh}, \tag{5.3}$$

where B^{i}_{ijh} denotes the ordinary curvature tensor of the symmetric connection Γ and we indicate by a semi-colon covariant differentiation with respect to the Γ.

Having into account the known identities

$$B^{i}_{ijh}+B^{i}_{ihj} = 0, \quad B^{i}_{ijh}+B^{i}_{jhi}+B^{i}_{hij} = 0 \tag{5.4}$$

from (5.3) we deduce

$$R^{i}_{ijh}(a, a'; a)+R^{i}_{ihj}(a, a'; a) = 2(\Omega^{i}_{ih;j}+\Omega^{i}_{ij;h}) \tag{5.5}$$

$$R^{i}_{ijh}(a, a'; a)+R^{i}_{jhi}(a, a'; a)+R^{i}_{hij}(a, a'; a) \tag{5.6}$$

$$= 4(\Omega^{i}_{mj}\Omega^{m}_{hi}+\Omega^{i}_{mh}\Omega^{m}_{ij}+\Omega^{i}_{mi}\Omega^{m}_{jh}).$$

When $R^{i}_{ijh}(a, a'; a) = 0$, from (5.5), (5.6) and (5.3) we deduce

$$\Omega^{i}_{ih;j}+\Omega^{i}_{ij;h} = 0, \qquad B^{i}_{ijh} = \Omega^{i}_{im}\Omega^{m}_{jh}. \tag{5.7}$$

If, moreover, $R^{i}_{ijh}(a, a; a)=0$, an analogous computation shows that $\Omega^{i}_{ih;j}-\Omega^{i}_{ij;h}=0$. Consequently we have $\Omega^{i}_{ih;j} = 0$ and a known theorem of Eisenhart (Eisenhart, 1933, p. 197) may be stated in the following terms :

Theorem. *A necessary and sufficient condition that an asymmetric connection* $^{a}\Gamma$ *determines a simply transitive Lie group is that the equations*

$$R^{i}_{ijh}(a, a; a) = 0, \qquad R^{i}_{ijh}(a, a'; a) = 0 \tag{5.8}$$

be satisfied.

Other tensors which may be useful are the following

$$R^{i}_{ijh}(a, a'; a') = R^{i}_{ijh}(a, a'; a)+4\Omega^{i}_{im}\Omega^{m}_{jh} \tag{5.9}$$

$$R^{i}_{ijh}(a', a; a) = -R^{i}_{ihj}(a, a'; a') \tag{5.10}$$

$$R^{i}{}_{ijh}(a', a; a') = -R^{i}{}_{ihj}(a, a'; a) \tag{5.11}$$

If we represent by $^{0}\Gamma^{i}_{ij}$ the symmetric connection Γ^{i}_{ij}, by direct computation we get

$$R^{i}_{ijh}(a, 0; a) = B^{i}_{ijh}+\Omega^{i}_{ih;j}-\Omega^{i}_{im}\Omega^{m}_{jh} \tag{5.12}$$

$$R^{i}_{ijh}(a, 0; a') = B^{i}_{ijh}+\Omega^{i}_{ih;j}+\Omega^{i}_{im}\Omega^{m}_{j\,h} \tag{5.13}$$

$$R^{i}_{ijh}(0, a; a) = B^{i}_{ijh}-\Omega^{i}_{ij;h}+\Omega^{i}_{im}\Omega^{m}_{jh} \tag{5.14}$$

$$R^{i}_{ijh}(0, a; a') = B^{i}_{ijh}-\Omega^{i}_{ij;h}-\Omega^{i}_{im}\Omega^{m}_{jh} \tag{5.15}$$

$$R^{i}_{ijh}(a, 0; 0) = B^{i}_{ijh}+\Omega^{i}_{ih;j} \tag{5.16}$$

$$R^{i}_{ijh}(0, a; 0) = B^{i}_{ijh}-\Omega^{i}_{ij;h} \tag{5.17}$$

Consequently, we have :

Theorem. *All generalized curvature tensors* (3.2) *which may be obtained with the connections* ${}^a\Gamma$, ${}^{a'}\Gamma$, ${}^0\Gamma = \Gamma$ *are linear combination of the tensors*

$$B^s_{ijh}, \quad \Omega^s_{ih;j}, \quad \Omega^s_{im}\Omega^m_{jh}. \tag{5.18}$$

6. Another Example. Let us consider the case of two connections ${}^a\Gamma$, ${}^b\Gamma$ which define the same parallelism in the space. Then we have

$${}^b\Gamma^s_{ij} = {}^a\Gamma^s_{ij} + \delta^s_i\psi_j \tag{6.1}$$

where ψ_j is an arbitrary covariant vector (Eisenhart, 1927, p. 30).

In this case we have

$$R^s_{ijh}(a, b; c) = R^s_{ijh}(a) - \delta^s_i\psi_{j;h}(c) \tag{6.2}$$

$$R^s_{ijh}(a, a; c) = R^s_{ijh}(a) \tag{6.3}$$

$$R^s_{ijh}(b, b; c) = R^s_{ijh}(a) + \delta^s_i(\psi_{h;j} - \psi_{j;h}) \tag{6.4}$$

$$R^s_{ijh}(b, a; c) = R^s_{ijh}(a) + \delta^s_i\psi_{h;j}(c') \tag{6.5}$$

where we have indicated by $\psi_{h;j}(c)$ covariant differentiation with respect to ${}^c\Gamma$. The last identities show the following

Theorem. *All generalized curvature tensors* (3.2) *corresponding to connections which define the same parallelism, differ among them only by a linear combination of tensors of the form* $\delta^s_i\psi_{j;h}$ *where* ψ_j *is an arbitrary covariant vector and the covariant derivative is taken with respect to any one of the given connections.*

Finally let us consider the case of two *symmetric* connections which define the same paths. According to Eisenhart (Eisenhart, 1927, p. 56) and Schouten (Schouten, 1954, p. 156) the connections are then in the relation

$${}^b\Gamma^s_{ij} = {}^a\Gamma^s_{ij} + \delta^s_i\psi_j + \delta^s_j\psi_i \tag{6.6}$$

where ψ_i is an arbitrary covariant vector.

The tensors $R^s_{ijh}(a, b; c)$ (3.2) become

$$R^s_{ijh}(a, b; c) = R^s_{ijh}(a) - \delta^s_i\psi_{j;h} - \delta^s_j\psi_{i;h} + ({}^c\Gamma^m_{jh} - {}^a\Gamma^m_{jh})(\delta^s_i\psi_m + \delta^s_m\psi_i) \tag{6.7}$$

$$R^s{}_{ijh}(b, a; c) = R^s_{ijh}(a) + \delta^s_i\psi_{h;j} + \delta^s_h\psi_{i;j} - ({}^c\Gamma^m_{jh} - {}^a\Gamma^m_{jh})(\delta^s_m\psi_i + \delta^s_i\psi_m) \tag{6.8}$$

$$R^s_{ijh}(b, b; c) = R^s_{ijh}(a) + \delta^s_i(\psi_{h,j} - \psi_{j;h}) + \delta^s_h(\psi_{i;j} - \psi_i\psi_j) - \delta^s_j(\psi_{i;h} - \psi_i\psi_h) \tag{6.9}$$

where the covariant derivatives are with respect to the symmetric connection ${}^a\Gamma$.

From this result, by considering the cases $c = a, b$ we have :

Theorem. *All generalized curvature tensors* (3.2) *corresponding to two symmetric connections which define the same paths, differ among them by a linear combination of the tensors* $\delta^s_i\psi_{h;j}$, $\delta^s_i\psi_h\psi_j$, *where* ψ_i *is an arbitrary covariant vector and the covariant derivatives are with respect to* ${}^a\Gamma$.

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