INTEGRAL FORMULAS IN CROFTON'S STYLE ON THE SPHERE AND SOME INEQUALITIES REFERRING TO SPHERICAL CURVES

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Introduction. Several integral formulas referring to convex plane curves, notable for their great generality, were obtained by W. Crofton in 1868 and successive years from the theory of geometrical probability [6], [7], [8], [9], [10].

A direct and rigorous exposition of Crofton's principal results, adding some new formulas, was made in 1912 by H. Lebesgue [12]. Another systematic exposition of Crofton's most interesting formulas, together with the generalization of many of them to space, is found in the two volumes on integral geometry by Blaschke [2].

The purpose of the present paper is to give a generalization of Crofton's formulas to the surface of the sphere. This is what we do in part I. We find further integral formulas on the sphere (for instance, (16), (17), (20), (21)) which have no equivalent in the plane. Other formulas, if we consider the plane as the limit of a sphere whose radius increases indefinitely, give integral formulas referring to plane convex curves (e. g., (34), (35)) which we think are new.

In part II, with simple methods of integral geometry [2], we obtain three inequalities referring to spherical curves. Inequality (38) is the generalization to the sphere of an inequality that Hornich [11] obtained for plane curves. (52) and (58) contain the classical isoperimetric inequality on the sphere. Finally, inequality (61) gives a superior limitation for the "isoperimetric deficit" of convex curves on the sphere.

I. FORMULAS IN THE CROFTON STYLE ON THE SPHERE

1. Notation and useful formulas. The element of area on the sphere of unit radius will be represented by $d\Omega$; that is, if θ and φ are the spherical coördinates of the point Ω, we have

$$d\Omega = \sin\theta \, d\theta \, d\varphi. \tag{1}$$

A great non-directed circle C on the same sphere of unit radius can be determined by one of its poles, that is, by either of the extremities of the diameter perpendicular to it. Since $d\Omega$ is the element of area of one of these extremities, the "density" for measuring sets of great circles on the sphere is [2; 61, 80]

$$dC = d\Omega; \tag{2}$$

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that is, the "measure" of a set of great circles on the sphere is defined as the integral of (2) extended over this set.

It is possible to give the density (2) another form, which will sometimes be useful. We consider a fixed great circle C_0 and a fixed point A on it. The great circle C can be determined for the abscissa t of one of the intersection points from C and C_0 and the angle α between the two circles. If θ and φ are the spherical coördinates of the pole Ω of C with regard to the pole Ω_0 of C_0, $\theta = \alpha$, $\varphi = t$, and (1), (2) give

$$dC = \sin \alpha \, d\alpha \, dt. \tag{3}$$

Let us consider two great circles C_1, C_2 and one of their intersection points Ω. If α_1 and α_2 are the angles that C_1 and C_2 make with another fixed great circle which also passes through Ω, the following differential formula [2; 78] is known:

$$dC_1 \, dC_2 = | \sin (\alpha_2 - \alpha_1) | \, d\alpha_1 \, d\alpha_2 \, d\Omega. \tag{4}$$

By (2), formula (4) can be transformed into a "dual" form. Let Ω_1 and Ω_2 be two points on the unit sphere and let C be the great circle determined by them. If β_1 and β_2 are the abscissas of Ω_1 and Ω_2 on C in relation to a fixed origin on this circle, (4) is equivalent to

$$d\Omega_1 \, d\Omega_2 = | \sin (\beta_1 - \beta_2) | \, d\beta_1 \, d\beta_2 \, dC. \tag{5}$$

2. **First integral formulas. Convex curves on the sphere.** A closed curve on the sphere is said to be *convex* when it cannot be cut by a great circle in more than two points.

A convex curve divides the surface of the sphere into two parts, one of which is always wholly contained in a hemisphere; that is, there is always a great circle which has the whole convex curve on the same side; we only have to consider, for example, a tangent great circle.

When we say a "convex figure", we understand that part of the surface of the sphere which is limited by a convex curve and is smaller than or equal to a hemisphere.

Let us consider a convex figure K on the sphere of unit radius. The radii which are perpendicular to the tangent planes (or, more generally, to the planes of support) to the cone which projects K from the center of the sphere form another cone whose intersection with the sphere is a new convex curve K^*. We shall call K^* the "dual" curve of K. The lengths and areas of K and K^* are connected by the known relations

$$F^* = 2\pi - L, \qquad L^* = 2\pi - F. \tag{6}$$

All the great circles C that cut K have their poles in the area bounded by K^* and the symmetrical curve of the same K^* with respect to the center of the sphere. This area equals $4\pi - 2F^* = 2L$. Counting each pair of points which are the extremities of a diameter as a single point, and taking into account the value (2) of the density dC, we have

$$\int_{C \cdot K \neq 0} dC = L; \tag{7}$$

this means: on the sphere, the measure of the great circles which cut a convex curve is equal to the length of this curve. This result is given by [2; 81].

3. **Integral of the chords.** Let Ω_1 and Ω_2 be two points inside the convex curve K (always on the unit sphere) and let C be the great circle determined by them. The differential expression (5) can be integrated for all pairs of points within K.

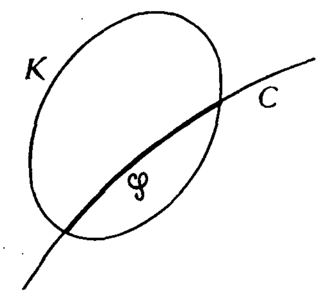

FIGURE 1

The integral of the left side is F^2. By calculating the integral of the right side, if φ represents the length of the arc of C that is contained in K (Fig. 1), we have

$$\int_0^{\varphi} \int_0^{\varphi} | \sin (\beta_1 - \beta_2) | \, d\beta_1 \, d\beta_2 = 2(\varphi - \sin \varphi). \tag{8}$$

Hence

$$\int_{C \cdot K \neq 0} (\varphi - \sin \varphi) \, dC = \tfrac{1}{2} F^2. \tag{9}$$

This formula generalizes, as we shall see (§11), Crofton's formula for chords in plane geometry.

4. **Principal Crofton formula.** Let us consider all the pairs of great circles C_1, C_2 that cut K. From (7) we deduce

$$\int_{\substack{C_1 \cdot K \neq 0 \\ C_2 \cdot K \neq 0}} dC_1 \, dC_2 = L^2. \tag{10}$$

Now we can make the integration of formula (4) extend only to the pairs of great circles which cut K. If Ω is fixed inside K, α_1 and α_2 can vary from 0 to π and

$$(11)\qquad \int_{\Omega\subset K}\left(\int_0^\pi\int_0^\pi |\sin(\alpha_1-\alpha_2)|\,d\alpha_1\,d\alpha_2\right)d\Omega = 2\pi\int_{\Omega\subset K} d\Omega = 2\pi F;$$

if Ω is outside K, α_1 and α_2 can vary from 0 to the angle ω between the great circles which are tangent to K and which pass through Ω (Fig. 2). By applying

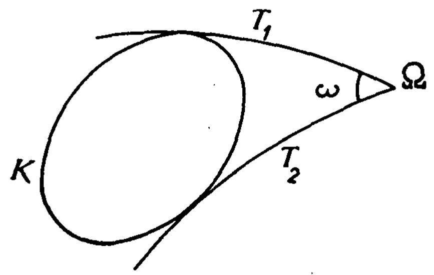

FIGURE 2

(8), the value of this last integral is found to be $\int 2(\omega - \sin\omega)\,d\Omega$ for $\Omega \not\subset K$. Adding this result to (11), we have (10); hence

$$(12)\qquad \int(\omega - \sin\omega)\,d\Omega = \tfrac{1}{2}L^2 - \pi F \qquad (\Omega \not\subset K).$$

This formula has the same form as Crofton's fundamental formula of plane geometry. The integration in (12) is extended to all points Ω outside K, each pair of points situated in the extremities of a diameter being considered as a single point.

5. **"Dual" formulas.** From a convex curve K we can deduce the "dual" curve K^* as we have seen in §2. To a great circle C which cuts K corresponds a point Ω^* (the pole of C) which is not inside K^*. The arc φ of C inside K is equal to $\pi - \omega^*$, ω^* being the angle between the two great circles tangent to K^* drawn through Ω^*. Since $F = 2\pi - L^*$ (by (6)), formula (9) can be written

$$(13)\qquad \int(\pi - \omega^* - \sin\omega^*)\,d\Omega^* = \tfrac{1}{2}(2\pi - L^*)^2 \qquad (\Omega^* \not\subset K^*).$$

The integration is extended over the outside of K^* (the points which are the extremities of the same diameter being considered as a single point) and consequently

$$\int \pi\,d\Omega^* = \pi(2\pi - F^*) \qquad (\Omega^* \not\subset K^*).$$

Then (13) gives

$$(14) \qquad \int (\omega^* + \sin \omega^*)\, d\Omega^* = 2\pi L^* - \pi F^* - \tfrac{1}{2} L^{*2} \qquad (\Omega^* \not\subset K^*).$$

This formula holds for any convex curve K^*; hence it is valid for K:

$$(15) \qquad \int (\omega + \sin \omega)\, d\Omega = 2\pi L - \pi F - \tfrac{1}{2} L^2 \qquad (\Omega \not\subset K).$$

From (15) and (12), we deduce

$$(16) \qquad \int \omega\, d\Omega = \pi L - \pi F \qquad (\Omega \not\subset K)$$

and

$$(17) \qquad \int \sin \omega\, d\Omega = \pi L - \tfrac{1}{2} L^2 \qquad (\Omega \not\subset K).$$

The same procedure shows that formula (12) is equivalent to

$$(18) \qquad \int_{C^* \cdot K^* \neq 0} (\pi - \varphi^* - \sin \varphi^*)\, dC^* = \tfrac{1}{2}(2\pi - F^*)^2 - \pi(2\pi - L^*),$$

where the integration is extended over all the great circles C^* which cut K^*. By (7) we have $\pi \int dC^* = \pi L^*$ and by substitution of this value in (18) and writing the formula for K, we have

$$(19) \qquad \int_{C \cdot K \neq 0} (\varphi + \sin \varphi)\, dC = 2\pi F - \tfrac{1}{2} F^2,$$

where φ is the length of the arc of C which is inside K.
From (9) and (19) we deduce

$$(20) \qquad \int_{C \cdot K \neq 0} \varphi\, dC = \pi F,$$

and

$$(21) \qquad \int_{C \cdot K \neq 0} \sin \varphi\, dC = \pi F - \tfrac{1}{2} F^2.$$

We repeat. In (16), (17), ω is the angle between the two great circles tangent to K through Ω; in (20), (21), φ is the length of the arc of the great circle C which is inside K.

The formulas (16), (17), (20), (21) that hold for any convex curve on the unit sphere have no equivalent in the plane.

6. **Formulas for the tangents.** Let K be a convex curve on the unit sphere with continuous radius of geodesic curvature.

If τ is the angle between a variable tangent great circle and a fixed tangent great circle and if s is the length of the arc of K, the radius of geodesic curvature ρ_g is given by

$$\rho_g = \frac{ds}{d\tau}, \tag{22}$$

and the Gauss-Bonnet formula gives

$$\oint_K \frac{ds}{\rho_g} = \int d\tau = 2\pi - F. \tag{23}$$

Let us consider two great circles tangent to K; let Ω be one of the intersection points of these circles. T_1 and T_2 will be the lengths of the arcs of these great circles bounded by Ω and the points of contact (T_1 and $T_2 \leq \pi$), and we represent by ω the angle between the two tangent circles at Ω (Fig. 2).

We wish to express the element of area $d\Omega$ as a function of the angles τ_1, τ_2 which determine the tangent great circles.

For fixed τ_2, as we pass from τ_1 to $\tau_1 + d\tau_1$, the arc T_2 is increased by $dT_2 = (\sin T_1/\sin \omega)d\tau_1$.

In the same way, as we pass from τ_2 to $\tau_2 + d\tau_2$, the arc T_1 is increased by $dT_1 = (\sin T_2/\sin \omega)d\tau_2$.

Since the element of area $d\Omega$ can be expressed in the form $d\Omega = \sin \omega \, dT_1 \, dT_2$, we find the desired expression

$$d\Omega = \frac{\sin T_1 \cdot \sin T_2}{\sin \omega} d\tau_1 \, d\tau_2$$

or

$$\frac{\sin \omega}{\sin T_1 \cdot \sin T_2} d\Omega = d\tau_1 \, d\tau_2 . \tag{24}$$

7. We can make the integration of (24) extend over all pairs of circles tangent to K and, by counting each pair once only (to do this we must divide the integral by 2), we have, by (23),

$$\int \frac{\sin \omega}{\sin T_1 \cdot \sin T_2} d\Omega = \tfrac{1}{2}(2\pi - F)^2 \qquad (\Omega \not\subset K). \tag{25}$$

Likewise, as in the preceding cases, the notation $\Omega \not\subset K$ indicates that the integration must be extended over all points Ω outside K; the points situated in the extremities of a diameter are considered as a single point.

8. Let $\rho_g^{(1)}, \rho_g^{(2)}$ be the radii of geodesic curvature of K at the points of contact of the tangent great circles through Ω. By virtue of (22), (24), we have

$$\sin \omega \frac{\rho_g^{(1)} \rho_g^{(2)}}{\sin T_1 \cdot \sin T_2} d\Omega = ds_1 ds_2 .$$

By integrating this expression over all pairs of tangent great circles, counting each pair once only, we get

$$\int \sin \omega \frac{\rho_g^{(1)} \rho_g^{(2)}}{\sin T_1 \cdot \sin T_2} d\Omega = \tfrac{1}{2}L^2 \qquad (\Omega \not\subset K). \tag{26}$$

9. By (22) and (24), we have

$$\sin \omega \frac{\rho_g^{(1)}}{\sin T_1 \cdot \sin T_2} d\Omega = ds_1 \, d\tau_2$$

and by integrating over all great circles tangent to K and observing that each point Ω is a common factor of two terms, it follows that

$$\int \sin \omega \frac{\rho_g^{(1)} + \rho_g^{(2)}}{\sin T_1 \cdot \sin T_2} d\Omega = L(2\pi - F) \qquad (\Omega \not\subset K). \tag{27}$$

10. **"Dual" formulas.** According to §5, from formulas (25), (26), (27) we can deduce the respective "dual" formulas.

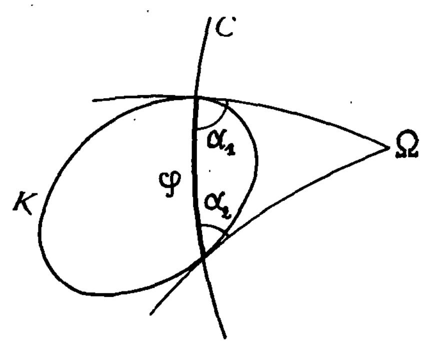

FIGURE 3

If φ is the length of the arc of the great circle C which is inside K and α_1, α_2 are the angles that C makes with the great circles tangent to K at the intersection points of C with K (Fig. 3), formula (25) gives

$$\int_{C \cdot K \neq 0} \frac{\sin \varphi}{\sin \alpha_1 \cdot \sin \alpha_2} dC = \tfrac{1}{2}L^2. \tag{28}$$

We observe that the "dual" element of ds is $d\tau^*$ for the dual curve K^* and reciprocally. Then the dual expression of $\rho_g = ds/d\tau$ will be $d\tau^*/ds^* = 1/\rho_g^*$. Hence, formula (26) gives

$$\int_{C \cdot K \neq 0} \frac{\sin \varphi}{\sin \alpha_1 \cdot \sin \alpha_2} \cdot \frac{1}{\rho_g^{(1)} \rho_g^{(2)}} \, dC = \tfrac{1}{2}(2\pi - F)^2. \tag{29}$$

Likewise, formula (27) gives

$$\int_{C \cdot K \neq 0} \frac{\sin \varphi}{\sin \alpha_1 \cdot \sin \alpha_2} \left(\frac{1}{\rho_g^{(1)}} + \frac{1}{\rho_g^{(2)}} \right) dC = (2\pi - F)L. \tag{30}$$

11. **Passage to the case of the plane.** The classical Crofton formulas for the plane must result as a special case of the preceding formulas when the radius of the sphere increases indefinitely. Moreover, by this procedure, we shall find some new integral formulas.

We observe the following. (i) The element of area $d\Omega$ on the unit sphere can be replaced by dP/R^2, where dP is the element of area on the sphere of radius R and, as $R \to \infty$, dP will be the element of area in the plane. (ii) Let us consider the form (3) for dC; for the sphere of radius R this expression (3) must be replaced by $dC = \sin \alpha \, d\alpha (dt_R/R)$, where t_R is the length of the arc of the great circle of the sphere of radius R; when R increases to ∞, (3) is lim $R \cdot dC = dG$, dG being the "density" of the straight lines of the plane (recall that the "density" dG can be written $dG = \sin \alpha \, d\alpha \, dt$, where α is the angle which G forms with another fixed straight line and t is the abscissa of the intersection point [2; 7]). (iii) When we consider a sphere of radius R, the area F and length L which are in formulas from §§2-10 must be replaced by F/R^2 and L/R, respectively.

When these remarks are taken into account, the preceding formulas give the following results.

(i) Let us consider formula (9). If σ is the length of the arc that the great circle C determines in K, then $\varphi = \sigma/R$ and for R large we have

$$\varphi - \sin \varphi = \frac{\sigma^3}{3!R^3} - \frac{\sigma^5}{5!R^5} + \cdots .$$

If dC and F are replaced in (9) by dG/R and F/R^2, as $R \to \infty$ we have

$$\int_{G \cdot K \neq 0} \sigma^3 \, dG = 3F^2. \tag{31}$$

This is the classical chord formula from Crofton [9; 84], [10; 27], [2; 20].

(ii) Formula (12) maintains the same form for the plane. Indeed, ω and $\sin \omega$ do not change; $d\Omega$ becomes dP/R^2, F becomes F/R^2, and L becomes L/R; in the limit as $R \to \infty$, formula (12) does not change. It is the "principal" Crofton formula for the plane [9; 78], [10; 26], [2; 18].

(iii) Formulas (16), (17), (20), (21) have no equivalent in the plane, since, when these formulas are written for the sphere of radius R, as $R \to \infty$ the right side increases indefinitely.

(iv) In formula (25), we must replace $\sin T_1$ and $\sin T_2$ by T_1/R and T_2/R, the element of area $d\Omega$ by dP/R^2, and F by F/R^2. In the limit as $R \to \infty$, we find

$$\int \frac{\sin \omega}{T_1 T_2}\, dP = 2\pi^2 \qquad (P \not\subset K). \tag{32}$$

In this well-known formula ([12]; see also W. Blaschke, *Differentialgeometrie* I, p. 49), T_1 and T_2 are the lengths of the tangents to the convex curve K drawn through P, $dP = dx\, dy$ is the element of area on the plane, and ω is the angle between the tangents at P.

For formulas (26), (27), it is only necessary to observe that the radii of geodesic curvature become the radii of the ordinary curvature of the plane curve. Hence formulas (26) and (27) give the known formulas [12]

$$\int \sin \omega \frac{\rho_1 \rho_2}{T_1 T_2}\, dP = \tfrac{1}{2} L^2, \qquad \int \sin \omega \frac{\rho_1 + \rho_2}{T_1 T_2}\, dP = 2\pi L \qquad (P \not\subset K). \tag{33}$$

(v) Formula (28), when φ is replaced by σ/R (σ is the length of the arc that the great circle C determines in K and in the limit it is the length of the chord that the straight line G determines in K) and R increases indefinitely, gives

$$\int_{G \cdot K \neq 0} \frac{\sigma}{\sin \alpha_1 \cdot \sin \alpha_2}\, dG = \tfrac{1}{2} L^2. \tag{34}$$

α_1 and α_2 are the angles that the straight line G makes with the tangents to K at the intersection points of G with K. The integration in (34) is extended over all the straight lines G which cut K.

Likewise, (29) and (30) give for the plane

$$\begin{aligned} &\int_{G \cdot K \neq 0} \frac{\sigma}{\rho_1 \rho_2 \sin \alpha_1 \sin \alpha_2}\, dG = 2\pi^2, \\ &\int_{G \cdot K \neq 0} \frac{(\rho_1 + \rho_2)\sigma}{\rho_1 \rho_2 \sin \alpha_1 \sin \alpha_2}\, dG = 2\pi L, \end{aligned} \tag{35}$$

where ρ_1 and ρ_2 are the radii of curvature of the convex curve K at the intersection points of G with K.

II. SOME INEQUALITIES REFERRING TO SPHERICAL CURVES

12. **A known formula.** Hitherto we have only considered relations on the sphere between a convex curve K and points and great circles. Now we wish to

establish some new relations which arise from considering on the sphere sets of variable small circles of constant spherical radius.

Let $\mathfrak{L}$ be a rectifiable curve (not necessarily convex) of length L on the unit sphere. We consider on the same sphere a small circle C_0 of spherical radius ρ $(\rho \leq \frac{1}{2}\pi)$, whose length and area will be

$$(36) \qquad L_0 = 2\pi \sin \rho, \qquad F_0 = 2\pi(1 - \cos \rho).$$

Let Ω be the center of the circle C_0 and, as in §1, $d\Omega$ the corresponding element of area of the sphere. If n represents the number of intersection points of the curve $\mathfrak{L}$ with the circle C_0 (n will be a function of Ω), we have the known formula

$$\int n \, d\Omega = \frac{2}{\pi} L L_0 ,$$

or

$$(37) \qquad \int n \, d\Omega = 4L \sin \rho;$$

the integration is extended over the whole sphere.

This formula is a particular case of Poincaré's formula of integral geometry [2; 81]. In [2], the formula is established only for spherical curves composed of a finite number of arcs with a continuously turning tangent. More generally, formula (37) is also valid for the case of a curve $\mathfrak{L}$ only supposed to be rectifiable and a circle C_0. The proof can be copied step by step from that given for Euclidean space of n dimensions in [13].

13. **An inequality referring to rectifiable curves on the sphere.** In this section we generalize for curves on the sphere an inequality that Hornich obtained for Euclidean space [11]. The proof is analogous to that given for Euclidean space in [13].

Let us consider on the sphere of unit radius the rectifiable curve $\mathfrak{L}$ of length L. Let F be the area filled by the points of the sphere whose spherical distance from $\mathfrak{L}$ is $\rho \leq \frac{1}{2}\pi$.

We shall prove that

$$(38) \qquad F \leq 2L \sin \rho + 2\pi(1 - \cos \rho)$$

and establish the conditions for the equality in (38).

Let M_i $(i = 0, 1, 2, 3, \cdots)$ be the area covered by the centers of the circles of radius ρ whose distance to $\mathfrak{L}$ is not greater than ρ and which have i points in common with $\mathfrak{L}$.

By (37), we have

$$(39) \qquad M_1 + 2M_2 + 3M_3 + 4M_4 + \cdots = 4L \sin \rho,$$

and according to the definition of the area F,

$$(40) \qquad M_0 + M_1 + M_2 + M_3 + \cdots = F.$$

From (39) and (40) we deduce

$$(41) \qquad 2F - 4L \sin \rho = 2M_0 + M_1 - (M_3 + 2M_4 + \cdots).$$

We consider the arc of a great circle of length D ($\leq \pi$) which joins the extremities of the given curve $\mathfrak{L}$ (if this curve is closed, $D = 0$). Let us call M_i^* ($i = 0, 1, 2$) the area covered by the centers of the circles of spherical radius ρ which have i points in common with this arc of length D (for $i = 0$ the arc is interior to the circle).

The area filled by the points whose distance from the arc of length D is not less than $\rho \leq \frac{1}{2}\pi$ is limited by two arcs of circles parallel to this arc at the distance ρ and two semicircles of radius ρ at the ends. The value of this area is $2D \sin \rho + 2\pi(1 - \cos \rho)$ and we can write

$$(42) \qquad M_0^* + M_1^* + M_2^* = 2D \sin \rho + 2\pi(1 - \cos \rho).$$

By (37) we have also

$$(43) \qquad M_1^* + 2M_2^* = 4D \sin \rho.$$

From (42) and (43) we deduce

$$(44) \qquad 2M_0^* + M_1^* = 4\pi(1 - \cos \rho).$$

We observe that if the circle C of radius ρ contains in its interior the curve $\mathfrak{L}$, it contains also the arc D. Hence $M_0 \leq M_0^*$. Likewise if C cuts $\mathfrak{L}$ in only one point, it has one of its extremities in the interior and the other in the exterior and so the arc D cuts the circle C also at only one point, that is to say, $M_1 \leq M_1^*$. It follows that, by (41) and (44),

$$\begin{aligned} 2F - 4L \sin \rho &\leq 2M_0^* + M_1^* - (M_3 + 2M_4 + \cdots) \\ &= 4\pi(1 - \cos \rho) - (M_3 + 2M_4 + \cdots); \end{aligned}$$

hence

$$(45) \qquad F + \tfrac{1}{2}(M_3 + 2M_4 + \cdots) \leq 2\pi(1 - \cos \rho) + 2L \sin \rho.$$

This inequality implies (38).

The equality in (38) will be verified only if $M_i = 0$ for $i \geq 3$ and moreover $M_0 = M_0^*$, $M_1 = M_1^*$. The condition $M_i = 0$ for $i \geq 3$ carries with it $M_1 = M_1^*$; since in the case when the circle C cuts in only one point the arc of the great circle which joins the extremities of $\mathfrak{L}$, it must cut $\mathfrak{L}$ in an odd number of points. Consequently, the conditions for equality are:

(i) $M_i = 0$ (for $i \geq 3$). The curve $\mathfrak{L}$ cannot be cut by the circle C in more than two points.

(ii) $M_0 = M_0^*$, that is to say, if the circle C contains in its interior the two extremities of the curve $\mathfrak{L}$, it contains also the whole curve.

In particular, if the given curve $\mathfrak{L}$ is closed, the equality in (38) is valid only in the case of reduction to a point.

14. **Isoperimetric inequality on the sphere.** Let K be a convex curve on the sphere of unit radius. We consider the exterior parallel curve to K at the distance $\rho \leq \frac{1}{2}\pi$. This curve cannot have double points and its area is easy to calculate. The area is [3; 81]

$$S = F + L \sin \rho + 2\pi(1 - \cos \rho) - F(1 - \cos \rho), \tag{46}$$

or, with the values (36) of the area and the length of the circle of radius ρ,

$$S = F + F_0 + \frac{1}{2\pi}(LL_0 - FF_0). \tag{47}$$

Let us put, as in the last section, M_i $(i = 0, 2, 4, 6, \cdots)$ for the area covered by the centers of the circles of radius ρ which have i points in common with K (M_0 will be the area covered by the centers of the circles of radius ρ each of which contains K in its interior or which is contained in the interior of K). Since K is a closed curve, i is always even.

The expression (47) is equivalent to

$$M_0 + M_2 + M_4 + \cdots = F + F_0 + \frac{1}{2\pi}(LL_0 - FF_0) \tag{48}$$

and formula (37) gives

$$M_2 + 2M_4 + 3M_6 + \cdots = \frac{1}{\pi} LL_0 . \tag{49}$$

Let us consider a radius ρ such that $M_0 = 0$, that is, such that the circle of radius ρ neither can be totally interior to K nor can contain K in its interior. From (48) and (49) we deduce then

$$M_4 + 2M_6 + \cdots = \frac{1}{2\pi}(LL_0 + FF_0) - (F + F_0). \tag{50}$$

We observe that, by (36), $L_0^2 + F_0^2 - 4\pi F_0 = 0$; hence we can write the identity

$$\begin{aligned} &\frac{1}{2\pi}(LL_0 + FF_0) - (F + F_0) \\ &\qquad = \frac{1}{4\pi}[(L^2 + F^2 - 4\pi F) - (L - L_0)^2 - (F - F_0)^2] \end{aligned} \tag{51}$$

and (50) gives

$$L^2 + F^2 - 4\pi F = (L - L_0)^2 + (F - F_0)^2 + 4\pi(M_4 + 2M_6 + \cdots). \tag{52}$$

Since the second member of this equality always ≥ 0, we obtain the classical isoperimetric inequality on the sphere

$$L^2 + F^2 - 4\pi F \geq 0. \tag{53}$$

This inequality has often been proved. See [1], [3] and [2], and the bibliography in [4; 113]. For proof with methods of integral geometry analogous to those we follow in this paper, see [2; 83].

Equality (52) is valid when F_0 and L_0 are the area and length of any circle which neither contains K in its interior nor is contained in the interior of K. In particular, if C_0^* is the smallest circle which contains K in its interior and C_0 is the greatest circle which is contained in K, by neglecting the non-negative sum $M_4 + 2M_6 + \cdots$, we have

$$L^2 + F^2 - 4\pi F \geq (L - L_0)^2 + (F - F_0)^2, \tag{54}$$

$$L^2 + F^2 - 4\pi F \geq (L_0^* - L)^2 + (F_0^* - F)^2. \tag{55}$$

Taking into account the general inequality

$$x^2 + y^2 \geq \tfrac{1}{2}(x + y)^2, \tag{56}$$

we may combine inequalities (54) and (55) into the inequality

$$L^2 + F^2 - 4\pi F \geq \left(\frac{L_0^* - L_0}{2}\right)^2 + \left(\frac{F_0^* - F_0}{2}\right)^2. \tag{57}$$

This is a better form than (53) for the isoperimetric inequality.

If we substitute for L_0, L_0^*, F_0, F_0^* their values (36), relation (57) gives

$$L^2 + F^2 - 4\pi F \geq 4\pi^2 \sin^2 \frac{r_M - r_m}{2}, \tag{58}$$

where r_M and r_m are the spherical radii of the circles C_0^* and C_0.

T. Bonnesen [3; 82] has obtained the inequality

$$L^2 + F^2 - 4\pi F \geq 4\pi^2 \tan^2 \frac{r_M - r_m}{2},$$

which is better than our (58). His proof is completely different from ours.

For a sphere of radius R, inequality (57) takes the form

$$L^2 - 4\pi F + \left(\frac{F}{R}\right)^2 \geq \left(\frac{L_0^* - L_0}{2}\right)^2 + \left(\frac{F_0^* - F_0}{2R}\right)^2, \tag{59}$$

which as $R \to \infty$ gives the inequality

$$L^2 - 4\pi F \geq \left(\frac{L_0^* - L_0}{2}\right)^2 = \pi^2(r_M - r_m)^2, \tag{60}$$

which is a well-known isoperimetric inequality for plane curves established by Bonnesen [3; 63], [4; 113].

15. **An upper limitation for the isoperimetric deficit of convex spherical curves.** We now consider only convex curves K with *continuous radius of spherical curvature*. We understand by radius of spherical curvature the limit of the spherical radius of the circle which has three points in common with the curve as these points approach coincidence. This radius ρ ($\rho \leq \frac{1}{2}\pi$) is connected with the radius of geodesic curvature ρ_g by

$$\rho_g = \tan \rho.$$

Let ρ_M be the greatest radius and ρ_m the smallest radius of spherical curvature (both $\leq \frac{1}{2}\pi$). We wish to prove that

$$L^2 + F^2 - 4\pi F \leq \left(\frac{L_0^* - L_0}{2} + \frac{F_0^* - F_0}{2}\right)^2, \tag{61}$$

where L_0, F_0, L_0^*, F_0^* are now the lengths and areas of the circles whose radii are ρ_m and ρ_M respectively.

Likewise, as the area of the *exterior* parallel curve to K at distance ρ was expressed by (46), when we consider the *interior* parallel curve to K at a distance $\rho \leq \rho_m$, this curve will not have double points and its area is equal to

$$- L \sin \rho + F \cos \rho + 2\pi(1 - \cos \rho). \tag{62}$$

If we take $\rho = \rho_m$, area (62) will be the area covered by the centers of the circles of radius ρ_m which are contained in the interior of the convex curve K. If we represent this area by M_0, we can write

$$M_0 = - L \sin \rho_m + F \cos \rho_m + 2\pi(1 - \cos \rho_m). \tag{63}$$

We now wish to find the value of the area covered by the centers of the circles of radius ρ_M each of which contains K entirely in its interior. For this purpose we note that when the circle of radius ρ_M contains K in its interior, by a "dual" transformation (§2) the transformed circle (of radius $\frac{1}{2}\pi - \rho_M$) will be contained in the interior of the transformed curve K^* (whose length and area are $2\pi - F$ and $2\pi - L$ respectively). The area covered by the centers of the circles of radius ρ_M each of which contains K in its interior will then be given by (62) if we substitute ρ for $\frac{1}{2}\pi - \rho_M$, F for $2\pi - L$, and L for $2\pi - F$.

It follows that this area is given by

$$M_0^* = -L \sin \rho_M + F \cos \rho_M + 2\pi(1 - \cos \rho_M). \tag{64}$$

This has the same form as (63).

Let L_0, F_0 and L_0^*, F_0^* be the lengths and areas of the circles of radius ρ_m and ρ_M respectively, given by (36). Formulas (63) and (64) take the form

$$M_0 = F + F_0 - \frac{1}{2\pi}(LL_0 + FF_0) \tag{65}$$

and

$$M_0^* = F + F_0^* - \frac{1}{2\pi}(LL_0^* + FF_0^*). \tag{66}$$

When we take into account identity (51), these equalities give

$$L^2 + F^2 - 4\pi F = (L - L_0)^2 + (F - F_0)^2 - 4\pi M_0\,, \tag{67}$$

$$L^2 + F^2 - 4\pi F = (L_0^* - L)^2 + (F_0^* - F)^2 - 4\pi M_0^*. \tag{68}$$

Since M_0 and M_0^* are non-negative, we have

$$L^2 + F^2 - 4\pi F \le (L - L_0)^2 + (F - F_0)^2, \tag{69}$$

$$L^2 + F^2 - 4\pi F \le (L_0^* - L)^2 + (F_0^* - F)^2. \tag{70}$$

These inequalities give a first upper limit for the isoperimetric deficit $L^2 + F^2 - 4\pi F$.

From inequalities (69) and (70) we find

$$L^2 + F^2 - 4\pi F \le (L - L_0 + F - F_0)^2. \tag{71}$$

$$L^2 + F^2 - 4\pi F \le (L_0^* - L + F_0^* - F)^2. \tag{72}$$

Since the left sides are non-negative by (53) and since

$$xy \le \left(\frac{x + y}{2}\right)^2,$$

by multiplication of (71) and (72), we find

$$L^2 + F^2 - 4\pi F \le \left(\frac{L_0^* - L_0}{2} + \frac{F_0^* - F_0}{2}\right)^2. \tag{73}$$

For a sphere of radius R we have

$$L^2 - 4\pi F + \frac{F^2}{R^2} \le \left(\frac{L_0^* - L_0}{2} + \frac{F_0^* - F_0}{2R}\right)^2, \tag{74}$$

and as $R \to \infty$,

$$L^2 - 4\pi F \le \tfrac{1}{4}(L_0^* - L_0)^2 = \pi^2(\rho_M - \rho_m)^2, \tag{75}$$

where ρ_M and ρ_m are the greatest and the smallest radii of curvature of the plane convex curve K of length L and area F.

This inequality (75) is a known inequality obtained by Bottema [5]; see also [4; 83].

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