AFFINE INVARIANTS OF CERTAIN PAIRS OF CURVES AND SURFACES

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1. **Introduction.** For two curves in a plane or two surfaces in ordinary space various projective invariants have been given by Mehmke, Bouton, Segre, Buzano, Bompiani, Hsiung and others (see Bibliography).

Obviously each projective invariant is also an affine invariant, that is, an invariant with respect to the group of affine transformations. However in certain cases there are affine invariants which are not projective invariants. The purpose of the present paper is to study these cases giving affine invariants, as well as their affine and metrical characterization, for the following cases:

(a) two curves in a plane having a common tangent at two ordinary points (§§2, 3);

(b) two curves in a plane intersecting at an ordinary point (§§4, 5);

(c) two surfaces in ordinary space having a common tangent plane at two ordinary points (§§6, 7);

(d) two surfaces in ordinary space having a common tangent line but distinct tangent plane at two ordinary points (§§8, 9).

For the cases (a), (b) of plane curves we shall consider the neighborhoods of the second and the third order of the curves at the considered points. For the cases (c), (d) of two surfaces in ordinary space we shall consider only the neighborhoods of the second order of the surfaces at the considered points.

2. **Affine invariants of two plane curves having a common tangent at two ordinary points.** Suppose that O and O_1 are two ordinary points of two plane curves C and C_1 respectively, so that OO_1 is the common tangent. Let h be the distance OO_1. If we choose a cartesian coordinate system in such a way that the point O be the origin and the line OO_1 be the x-axis, the power series expansions of the two curves in the neighborhood of the points O and O_1 may be written in the form

$$(2.1) \qquad C: \qquad y = ax^2 + bx^3 + \cdots,$$

$$(2.2) \qquad C_1: \qquad y = a_1(x-h)^2 + b_1(x-h)^3 + \cdots,$$

where we suppose $a, a_1 \neq 0$.

In order to find the affine invariants of the elements of the second and the third order of the curves C, C_1 in the neighborhood of O, O_1 we have to consider the most general affine transformation which leaves the point O and the x-axis invariant:

$$(2.3) \qquad x = \alpha X + \beta Y, \qquad y = \mu Y,$$

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where α, β, μ are arbitrary constants. By this transformation the point O_1(is carried into the point whose coordinates are h/α, 0; consequently if we ca the distance between the transformed points from O and O_1 we have

$$\alpha = h/H. \tag{2.4}$$

Let us substitute (2.3) in (2.1) and (2.2). We find two equations of the

$$Y = AX^2 + BX^3 + \cdots, \qquad Y = A_1(X - H)^2 + B_1(X - H)^3 + \cdots$$

where the first coefficients A, B, A_1, B_1 are given by the following system:

$$\begin{aligned} \mu A &= \alpha^2 a, \qquad \mu B = 2\alpha\beta a A + \alpha^3 b, \\ \mu A_1 &= \alpha^2 a_1, \qquad \mu B_1 = 2\alpha\beta a_1 A_1 + \alpha^3 b_1. \end{aligned} \tag{2.5}$$

Eliminating α, β, μ from this system and (2.4) we find that *there are two i pendent affine invariants determined by the neighborhoods of the second and third order of the curves C and C_1 at the points O, O_1, which are*

$$I_1 = a/a_1, \qquad I_2 = ha(b_1/a_1^2 - b/a^2). \tag{2.6}$$

By interchanging a, b and a_1, b_1 in the invariant I_2 we find the invar $I_3 = -I_2/I_1$ which is a consequence of I_1 and I_2.

The invariant I_1 is determined by the neighborhoods of the second orde C, C_1 at the points O, O_1 and the invariant I_2 is determined by the neighborho of the third order.

3. **Metrical and affine characterization of the invariants I_1 and I_2.** If coordinate system to which the expansions (2.1) and (2.2) of C and C_1 referred is chosen orthogonal, it is easily verified that the radius of curva r of C and its derivative $r' = dr/ds$ with respect to the arc length s of C at point O are given by the formulas

$$r = \frac{1}{2a}, \qquad r' = -\frac{3b}{2a^2}. \tag{3.1}$$

Similarly the radius of curvature r_1 of C_1 and its derivative $r_1' = dr_1/ds_1$ w respect to the arc length s_1 of C_1 at the point O_1 are

$$r_1 = \frac{1}{2a_1}, \qquad r_1' = -\frac{3b_1}{2a_1^2}. \tag{3.2}$$

From (3.1), (3.2) we deduce that the invariants (2.6) are expressed metrica by the formulas

$$I_1 = \frac{r_1}{r}, \qquad I_2 = \frac{h}{3r}(r' - r_1'). \tag{3.3}$$

In order to give an affine characterization of the invariant I_1 let us cut curves C and C_1 by the parallel line to OO_1 given by the equation $y = \epsilon$. (

suppose $a > 0$; in case $a < 0$ we take $y = -\epsilon$.) If C and C_1 in the neighborhood of O and O_1 are situated at different sides of the line OO_1 we consider the two parallel lines $y = \pm\epsilon$. The area f bounded by this parallel line and the curve C in the neighborhood of O has the value

$$f = \frac{4}{3} a^{-\frac{1}{2}} \epsilon^{3/2} + o(\epsilon^{3/2}) \tag{3.4}$$

where $o(\epsilon^{3/2})$ means a function of ϵ such that $o(\epsilon^{3/2})/\epsilon^{3/2}$ tends to zero with ϵ.

Similarly the area f_1 bounded by the same line $y = \epsilon$ (or the symmetrical one $y = -\epsilon$) and the curve C_1 in the neighborhood of O_1 has the value

$$f_1 = \frac{4}{3} a_1^{-\frac{1}{2}} \epsilon^{3/2} + o(\epsilon^{3/2}). \tag{3.5}$$

From (3.4) and (3.5) we deduce

$$I_1 = a/a_1 = \lim_{\epsilon \to 0} (f_1/f)^2. \tag{3.6}$$

Since the quotient of two areas is an invariant with respect to affine transformations, (3.6) gives an affine interpretation of the invariant I_1.

Here let us make the following remark. It is known that the ratio between two areas situated on parallel planes is an invariant with respect to affine transformations in ordinary space. Consequently, if we consider two plane curves situated on parallel planes which have parallel tangents at two ordinary points O, O_1 by considering a plane which intersects both curves at a distance ϵ from the respective tangent, from (3.4) and (3.5) we deduce that *the ratio between the radii of curvature at O and O_1 of two curves situated on parallel planes with parallel tangents at O and O_1 respectively, is invariant with respect to affine transformations of the space.* This property will be used in §7.

In order to give an affine characterization of the invariant I_2 let us consider the osculating parabola of C at the point O. The equation of this parabola is found to be

$$y = ax^2 + \frac{b}{a} xy + \frac{b^2}{4a^3} y^2. \tag{3.7}$$

The diameter of this parabola which passes through O is called the *affine normal* of C at O [13; 49] and is the line

$$y = -\frac{2a^2}{b} x. \tag{3.8}$$

Similarly the affine normal of C_1 at O_1 is the line

$$y = -\frac{2a_1^2}{b_1} (x - h). \tag{3.9}$$

The polar line of the point O_1 with respect to the osculating parabola (3.7

$$2a^2hx + (bh - a)y = 0. \tag{3.10}$$

The intersection point M of the affine normal (3.9) of C_1 and the affine no (3.8) of C has the abscissa

$$\xi_0 = \frac{hb}{a^2(b/a^2 - b_1/a_1^2)}, \tag{3.11}$$

and the intersection point N of the affine normal (3.9) of C_1 and the polar (3.10) of O_1 with respect to the osculating parabola (3.7) has the abscissa

$$\xi_1 = \frac{(bh - a)h}{ha^2(b/a^2 - b_1/a_1^2) - a}. \tag{3.12}$$

Hence, taking into account the value (2.6) of I_2 we see that the r $D = NO_1/NM$ has the value

$$D = \frac{h - \xi_1}{\xi_0 - \xi_1} = -I_2 .$$

This formula gives an affine characterization of the invariant I_2, that *the affine invariant I_2 is equal, except for sign, to the ratio of the segments that affine normal of C at O, the polar line of O_1 with respect to the osculating para of C at O and the tangent OO_1 determine on the affine normal of C_1 at O_1.*

From this result we deduce that $I_2 = 0$ is the necessary and sufficient co tion that the affine normals of C and C_1 at O and O_1 respectively are para lines.

4. **Affine invariants of two curves in a plane intersecting at an ordinary po** Let us consider two plane curves C, C_1 intersecting at a point O and hav distinct tangents t and t_1 at O. Let us suppose that the point O is an ordin point on both C and C_1 and choose a cartesian coordinate system so that the x-axis and t_1 is the y-axis. Then the curves C and C_1 may be represen by expansions of the form

$$C: \qquad y = ax^2 + bx^3 + \cdots , \tag{4.1}$$

$$C_1: \qquad x = a_1y^2 + b_1y^3 + \cdots , \tag{4.2}$$

where a, $a_1 \neq 0$.

The most general affine transformation which leaves the point O and tangents t, t_1 invariant is represented by the equations

$$x = \alpha X, \qquad y = \mu Y, \tag{4.3}$$

where α, μ are arbitrary constants.

By means of this transformation the equations (4.1) and (4.2) of the curves C and C_1 transform into two other equations of the same form, namely,

$$Y = AX^2 + BX^3 + \cdots, \qquad X = A_1Y^2 + B_1Y^3 + \cdots,$$

where the coefficients A, B, A_1, B_1 are given by the following system:

$$(4.4) \qquad \begin{aligned} \mu A &= \alpha^2 a, & \mu B &= \alpha^3 b, \\ \alpha A_1 &= \mu^2 a_1, & \alpha B_1 &= \mu^3 b_1. \end{aligned}$$

Eliminating α, μ from these equations we find that *there are two independent affine invariants determined by the neighborhood of the third order of the curves C and C_1, namely,*

$$(4.5) \qquad i_1 = \frac{a_1 b^2}{a^3 b_1}, \qquad i_2 = \frac{a b_1^2}{a_1^3 b}.$$

Instead of these invariants it is more convenient to introduce the following ones:

$$(4.6) \qquad I_1^* = i_1^2 i_2 = \frac{b^3}{a_1 a^5}, \qquad I_2^* = i_1 i_2^2 = \frac{b_1^3}{a a_1^5}.$$

5. **Metric and affine characterization of the invariants I_1^* and I_2^*.** Let us call ω the angle which forms the tangents t, t_1 of the curves C, C_1 at the point O. The radius of curvature r of C and its derivative $r' = dr/ds$ with respect to the arc length s at the point O are found, on performing simple calculation, to be given by the formulas

$$(5.1) \qquad r = \frac{1}{2a \sin \omega}, \qquad r' = -\frac{3(b - 2a^2 \cos \omega)}{2a^2 \sin \omega},$$

from which we deduce

$$(5.2) \qquad a = \frac{1}{2r \sin \omega}, \qquad b = \frac{3 \cos \omega - r' \sin \omega}{6r^2 \sin^2 \omega}.$$

By analogy if r_1 and r_1' are the radius of curvature and its derivative with respect to the arc length s_1 of C_1 at the point O_1 we find

$$(5.3) \qquad a_1 = \frac{1}{2r_1 \sin \omega}, \qquad b_1 = \frac{3 \cos \omega - r_1' \sin \omega}{6r_1^2 \sin^2 \omega}.$$

From (5.2) and (5.3) we deduce that the affine invariants (4.6) have the following metric characterization:

$$(5.4) \qquad I_1^* = \frac{8}{27} \frac{r_1}{r} (3 \cos \omega - r' \sin \omega)^3, \qquad I_2^* = \frac{8}{27} \frac{r}{r_1} (3 \cos \omega - r_1' \sin \omega)^3.$$

In order to give an affine characterization of I_1^* and I_2^* we shall conside osculating parabola Q of C at the point O, which in the present case of ob axis is given also by the equation (3.7) and the osculating parabola Q_1 of the point O, given by the equation

$$(5.5) \qquad x = a_1 y^2 + \frac{b_1}{a_1} xy + \frac{b_1^2}{4a_1^3} x^2.$$

On the tangent t_1 of C_1 at O (y-axis of our coordinate system) we hav point A_1 in which t_1 intersects Q and the pole A_2 of the tangent t with res to the osculating parabola Q_1. An easy calculation proves that the dista from A_1 and A_2 to the point O are

$$(5.6) \qquad OA_1 = \frac{4a^3}{b^2}, \qquad OA_2 = \frac{a_1}{b_1},$$

respectively.

Similarly we have on the tangent t (x-axis of our coordinate system) the in section point B_1 of t with the osculating parabola Q_1 and the pole B_2 of t_1 respect to the osculating parabola Q, whose distances to the point O are

$$(5.7) \qquad OB_1 = \frac{4a_1^3}{b_1^2}, \qquad OB_2 = \frac{a}{b}.$$

Since the ratio in which a point divides a line segment is an invariant respect to the group of affine transformations, from (5.6) and (5.7) we de that the ratios

$$(5.8) \qquad \rho_1 = \frac{OA_2}{OA_1} = \frac{a_1 b^2}{4a^3 b_1}, \qquad \rho_2 = \frac{OB_2}{OB_1} = \frac{ab_1^2}{4ba_1^3}$$

are affine invariants. The invariants I_1^* and I_2^* are expressed by means o and ρ_2 by the formulas

$$(5.9) \qquad I_1^* = 64\rho_1^2\rho_2, \qquad I_2^* = 64\rho_1\rho_2^2$$

which give an affine characterization of I_1^* and I_2^*.

We conclude with the remark that the condition $I_1^* = 0$, which is equiva to $b = 0$, signifies that the affine normal of C at the point O (which in the pre case of oblique axis is given also by the equation (3.8)) coincides with the y-a that is, with the tangent t_1 of C_1 at O. Similarly, the condition $I_2^* = 0$ wh is equivalent to $b_1 = 0$, signifies that the affine normal of C_1 at O coincides w the x-axis, that is, with the tangent of C at O.

6. **Affine invariants of two surfaces in ordinary space having a comm tangent plane at two ordinary points.** Let S and S_1 be two surfaces in ordin space having a common tangent plane at two ordinary points O and O_1. Le be the distance OO_1. If we choose a cartesian coordinate system in such a w that the point O be the origin, the line OO_1 be the x-axis and the common tang

plane to S, S_1 at O, O_1 be the plane $z = 0$, the power series expansions of the two surfaces in the neighborhood of O and O_1 may be written in the form

$$(6.1) \qquad S: \quad z = ax^2 + bxy + cy^2 + \cdots,$$

$$(6.2) \qquad S_1: \quad z = a_1(x - h)^2 + b_1(x - h)y + c_1y^2 + \cdots.$$

In order to find the affine invariants determined by the neighborhoods of the second order of S, S_1 at the points O, O_1 we have to consider the most general affine transformation which leaves the point O, the x-axis and the plane $z = 0$ invariant, namely,

$$(6.3) \qquad \begin{aligned} x &= \alpha_1 X + \beta_1 Y + \gamma_1 Z \\ y &= \beta_2 Y + \gamma_2 Z \\ z &= \gamma_3 Z \end{aligned}$$

where α_1, β_1, γ_1, β_2, γ_2, γ_3 are arbitrary constants.

By means of this transformation the equations (6.1) and (6.2) of S and S_1 transform into two other equations of the same form, namely,

$$Z = AX^2 + BXY + CY^2 + \cdots,$$

$$Z = A_1(X - H)^2 + B_1(X - H)Y + C_1Y^2 + \cdots,$$

where $H = h/\alpha_1$ is the distance between the transformed points from O and O_1 and the coefficients are given by the following system:

$$(6.4) \qquad \begin{aligned} \gamma_3 A &= \alpha_1^2 a, & \gamma_3 A_1 &= \alpha_1^2 a_1, \\ \gamma_3 B &= 2\alpha_1\beta_1 a + \alpha_1\beta_2 b, & \gamma_3 B_1 &= 2\alpha_1\beta_1 a_1 + \alpha_1\beta_2 b_1, \\ \gamma_3 C &= \beta_1^2 a + \beta_1\beta_2 b + \beta_2^2 c, & \gamma_3 C_1 &= \beta_1^2 a_1 + \beta_1\beta_2 b_1 + \beta_2^2 c_1. \end{aligned}$$

According to the vanishing or non-vanishing of the two coefficients a and a_1 it is necessary to distinguish four cases:

Case I. $a \neq 0$, $a_1 \neq 0$. In this case the line OO_1 does not coincide with any one of the asymptotic tangents of the surfaces S, S_1 at O, O_1 respectively. Elimination of the coefficients α_1, β_1, γ_1, β_2, γ_2, γ_3 of the affine transformation (6.3) from equations (6.4) implies that *the affine invariants determined by the neighborhoods of the second order of the two surfaces S, S_1 at the points O, O_1 are the following*:

$$(6.5) \qquad J_{11} = \frac{a_1}{a}, \qquad J_{12} = \frac{4a_1c_1 - b_1^2}{4a_1^2(b_1/a_1 - b/a)^2}, \qquad J_{13} = \frac{4ac - b^2}{4a^2(b/a - b_1/a_1)^2}.$$

Case II. $a = 0$, $a_1 \neq 0$. In this case the line OO_1 is an asymptotic tangent of the surface S at O but does not coincide with an asymptotic tangent of S_1 at O_1. Elimination of the coefficients α_1, β_1, γ_1, β_2, γ_2, γ_3 from equations

(6.4) implies in this case that *the affine invariants determined by the neighborho of the second order of the surfaces S, S_1 at the points O, O_1 are the following*:

$$(6.6)\qquad J_{21} = \frac{1}{b^2}(bb_1 - 2ca_1), \qquad J_{22} = \frac{a_1}{b^4}(a_1c^2 - cbb_1 + b^2c_1).$$

Instead of J_{21} and J_{22} it is more convenient to replace J_{22} by the follow invariant

$$(6.7)\qquad \bar{J}_{22} = J_{21}^2 - 4J_{22} = \frac{b_1^2 - 4a_1c_1}{b^2}$$

so that as invariants for this case II we shall consider J_{21} and $\bar{J}_{22}$.

Case III. $a \neq 0$, $a_1 = 0$. In this case the line OO_1 is an asymptotic tang of S_1 at O_1 but is not an asymptotic tangent of S at O. Similarly, as in foregoing case, we obtain the invariants

$$(6.8)\qquad J_{31} = \frac{1}{b_1^2}(bb_1 - 2c_1a), \qquad \bar{J}_{32} = \frac{b^2 - 4ac}{b_1^2}.$$

Case IV. $a = 0$, $a_1 = 0$. In this case the line OO_1 is an asymptotic tang of both surfaces S and S_1 at O and O_1 respectively. Elimination of the efficients of the affine transformation (6.3) from equations (6.4) gives in case only one affine invariant, namely,

$$(6.9)\qquad J_{41} = \frac{b}{b_1}.$$

7. **Metric and affine characterization of the invariants J_{ij}.** For the purp of giving a metric characterization of the invariants J_{ij} of the foregoing numb let us suppose that the coordinate system to which the expansions (6.1) (6.2) are referred is an orthogonal cartesian coordinate system. Let us consi separately the four cases mentioned in the foregoing number.

Case I. $a \neq 0$, $a_1 \neq 0$. Let r and r_1 be the radii of curvature at the poi O and O_1 of the plane curves in which the normal plane $y = 0$ intersects w the surfaces S, S_1 respectively; and K and K_1 the total curvatures of S, at the points O, O_1. Then it can be readily demonstrated that

$$(7.1)\qquad r = \frac{1}{2a}, \qquad r_1 = \frac{1}{2a_1}, \qquad K = 4ac - b^2, \qquad K_1 = 4a_1c_1 - b_1^2.$$

Furthermore if ω is the angle which the conjugate direction at O of the tang OO_1 of the surface S forms with the tangent OO_1 and similarly ω_1 is the an between OO_1 and its conjugate direction at O_1 on the surface S_1, we have

$$(7.2)\qquad 2\cot\omega = -b/a, \qquad 2\cot\omega_1 = -b_1/a_1.$$

From (7.1), (7.2) and (6.5) we obtain the following metrical characterization of the affine invariants J_{11}, J_{12}, J_{13}:

$$(7.3)\qquad J_{11} = \frac{r}{r_1}, \qquad J_{12} = \frac{K_1 r_1^2}{4(\cot\omega_1 - \cot\omega)^2}, \qquad J_{13} = \frac{Kr^2}{4(\cot\omega - \cot\omega_1)^2}.$$

From these formulas it follows that the ratio of the total curvatures at O and O_1,

$$K/K_1 = J_{13}J_{12}^{-1}J_{11}^{-2},$$

is also an affine invariant of the two surfaces S, S_1. It may be remarked that the affine invariant $(K/K_1)(r/r_1)^{4/3} = J_{13}J_{12}^{-1}J_{11}^{-2/3}$ is also a projective invariant obtained by Hsiung [10].

We now proceed to give an affine characterization of the invariants J_{11}, J_{12}, J_{13}. We have seen (§3) that the ratio of the radii of curvature is an affine invariant for two plane curves with a common tangent line OO_1. Let us consider the plane curves C, C_1 in which the common normal plane to the surfaces S, S_1 through OO_1 intersects the surfaces S, S_1; let r, r_1 be the radii of curvature of C, C_1 at the points O, O_1. By an affine transformation the common normal plane to S, S_1 through the line OO_1 transforms into another plane which forms a certain angle, say θ, with the common normal plane to the transformed surfaces. If r^* and r_1^* are the radii of curvature of the transformed curves from C and C_1 at the transformed points from O, O_1, by §3, we have $r/r_1 = r^*/r_1^*$. Furthermore, if R and R_1 are the radii of curvature at the transformed points from O, O_1 of the curves in which the common, normal plane to the transformed surfaces from S, S_1 intersects these transformed surfaces, by Meusnier's theorem we have $r^* = R\cos\theta$, $r_1^* = R_1\cos\theta$. Consequently $r/r_1 = r^*/r_1^* = R/R_1$ and the affine invariance of $J_{11} = r/r_1$ is proved geometrically.

In order to find an affine characterization of the invariant J_{12}, let us consider the conjugate tangent t at O of the tangent OO_1 of the surface S. From the equation (6.1) of S we deduce that the equation of t is $y = -(2a/b)x$. On the other hand the asymptotic tangents of S_1 at O_1 are

$$(7.4)\qquad y = \frac{1}{2c_1}(-b_1 \pm (b_1^2 - 4a_1c_1)^{\frac{1}{2}})(x - h), \qquad z = 0.$$

A simple calculation shows that the tangent t intersects these asymptotic tangents in the points A_1, A_2 such that the invariant J_{12} can be expressed in terms of the ratio $\rho = OA_1/OA_2$ by the following formula

$$(7.5)\qquad J_{12} = -\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)^2$$

which gives an affine characterization of the invariant J_{12}.

A similar expression holds for the third affine invariant J_{13} expressed in terms of the ratio of the segments that the asymptotic tangents of S at O determine on the conjugate line at O_1 of the tangent O_1O of the surface S_1.

Case II. $a = 0, a_1 \neq 0$. In this case, with the same notations as i foregoing case, we have

$$(7.6) \qquad b^2 = -K, \qquad b_1 = -\frac{1}{r_1}\cot\omega_1, \qquad a_1 = \frac{1}{2r_1},$$

and if M means the mean curvature of S at O, since $a = 0$, it is known $c = M$. Hence the invariants J_{21} and $\bar{J}_{22}$ have the following metrical nificance:

$$(7.7) \qquad J_{21} = \frac{(-K)^{\frac{1}{2}} + M\tan\omega_1}{Kr_1\tan\omega_1}, \qquad \bar{J}_{22} = \frac{K_1}{K}.$$

We now proceed to give an affine characterization of these invariants. us consider the points P, Q in which the asymptotic tangent of S at O dist of OO_1 (given by the equations $y = -(b/c)x$, $z = 0$) intersects the asympt tangents of S_1 at O_1 (given by the equations (7.4)). A ready calculation sh that the ratio $\bar{J}_{22}/J_{21}^2$ is expressed in terms of the ratio $\rho = OP/OQ$ by formula

$$(7.8) \qquad \frac{\bar{J}_{22}}{J_{21}^2} = \left(\frac{1-\rho}{1+\rho}\right)^2.$$

On the other hand, let us consider the line t_1 (given by the equations y $-(2a_1/b_1)(x - h)$, $z = 0$), which is the harmonic conjugate of the comn tangent OO_1 with respect to the asymptotic tangents of the surface S_1 at point O_1; and its parallel line (given by the equations $y = -(2a_1/b_1)x$, $z =$ through the point O. The normal planes to S, S_1 through these parallel li intersect S and S_1 respectively in two plane curves whose radii of curvat at O, O_1 have the ratio

$$(7.9) \qquad \rho_1 = \frac{4a_1c_1 - b_1^2}{2(2a_1c - bb_1)} = \frac{\bar{J}_{22}}{2J_{21}}.$$

From (7.8) and (7.9) it follows that

$$J_{21} = 2\rho_1(1+\rho)^2(1-\rho)^{-2}, \qquad \bar{J}_{22} = 4\rho_1^2(1+\rho)^2(1-\rho)^{-2}.$$

These formulas give an affine characterization of the invariants J_{21} and $\bar{J}$

Case III. $a \neq 0, a_1 = 0$. This case is completely similar to the case II interchanging S and S_1.

Case IV. $a = 0, a_1 = 0$. In this case, according to (7.1) and (6.9) the or affine invariant can be written

$$(7.10) \qquad J_{41} = \left(\frac{K}{K_1}\right)^{\frac{1}{2}};$$

hence, the square of the affine invariant J_{41} is equal to the ratio of the tot curvatures of the surfaces S and S_1 at O and O_1 respectively.

To give an affine characterization of J_{41} let us consider the asymptotic tangent u of S at O distinct of OO_1 (given by the equations $y = -(b/c)x$, $z = 0$) and the asymptotic tangent u_1 of S_1 at O_1 distinct of OO_1 (given by the equations $y = -(b_1/c_1)(x - h)$, $z = 0$). These asymptotic tangents intersect each other in the point A whose coordinates are

$$(7.11) \qquad x = h(b_1/c_1)(b_1/c_1 - b/c)^{-1}, \qquad y = -h(b/c)(b_1/c_1)(b_1/c_1 - b/c)^{-1}.$$

We first suppose $b/c - b_1/c_1 \neq 0$. Let us consider the point B such that $2O_1B = O_1A$. The line OB is affinely connected with the configuration of the surfaces S, S_1. The normal plane to S at O through the line OB and the normal plane to S_1 at O_1 through the parallel line to OB through O_1 intersect S and S_1 respectively in two plane curves whose radii of curvature at O and O_1 have the ratio

$$(7.12) \qquad \frac{2b_1}{b} = 2J_{41}^{-1}.$$

Suppose now that $b/c - b_1/c_1 = 0$, so that the asymptotic tangents u and u_1 are parallel. In this case we may consider any pair of parallel normal planes through O and O_1 respectively; let $y = \lambda x$ and $y = \lambda(x - h)$ be these planes. The ratio between the radii of curvature at O and O_1 of the plane curves in which these planes intersect S and S_1 respectively, has the value

$$(7.13) \qquad \frac{b_1 + \lambda c_1}{b + \lambda c} = \frac{b_1}{b} = J_{41}^{-1}.$$

We have remarked in §3 that the ratio of the radii of curvature at two points O, O_1 of two plane curves situated on parallel planes with parallel tangent lines at the points O, O_1 is an invariant with respect to affine transformations of the space. Consequently (7.12) and (7.13) give an affine characterization of the invariant J_{41}.

8. **Affine invariants of two surfaces in ordinary space having a common tangent line but distinct tangent planes at two ordinary points.** Let now S and S_1 be two surfaces in ordinary space having a common tangent line but distinct tangent planes at two ordinary points O, O_1. Let h be the distance OO_1. If we choose a cartesian coordinate system in such a way that the point O be the origin, the tangent plane to S at O be the plane $z = 0$ and the tangent plane to S_1 at O_1 be the plane $y = 0$, the power series expansions of the two surfaces in the neighborhood of O and O_1 may be written in the form

$$(8.1) \qquad S: \qquad z = ax^2 + bxy + cy^2 + \cdots,$$

$$(8.2) \qquad S_1: \qquad y = a_1(x - h)^2 + b_1(x - h)z + c_1z^2 + \cdots.$$

In order to find the affine invariants determined by the neighborhoods of the second order of S and S_1 at the points O and O_1 we have to consider the most general affine transformation which leaves the point O and the plan $y = 0$, $z = 0$ invariants:

$$(8.3)\qquad \begin{aligned} x &= \alpha_1 X + \beta_1 Y + \gamma_1 Z \\ y &= \qquad\quad \beta_2 Y \\ z &= \qquad\qquad\quad \gamma_3 Z, \end{aligned}$$

where α_1, β_1, γ_1, β_2, γ_3 are arbitrary constants.

By means of this transformation the equations (8.1) and (8.2) transfor into two other equations of the same form, namely,

$$Z = AX^2 + BXY + CY^2 + \cdots,$$

$$Y = A_1(X - H)^2 + B_1(X - H)Z + C_1 Z^2 + \cdots,$$

where

$$(8.4)\qquad H = h/\alpha_1$$

is the distance between the transformed points from O and O_1, and the co efficients A, B, C, A_1, B_1, C_1 are given by the following system:

$$(8.5)\qquad \begin{aligned} \gamma_3 A &= \alpha_1^2 a, & \beta_2 A_1 &= \alpha_1^2 a_1, \\ \gamma_3 B &= 2\alpha_1\beta_1 a + \alpha_1\beta_2 b, & \beta_2 B_1 &= 2\alpha_1\gamma_1 a_1 + \alpha_1\gamma_3 b_1, \\ \gamma_3 C &= \beta_1^2 a + \beta_1\beta_2 b + \beta_2^2 c, & \beta_2 C_1 &= \gamma_1^2 a_1 + \gamma_1\gamma_3 b_1 + \gamma_3^2 c_1. \end{aligned}$$

According to the vanishing or non-vanishing of the two coefficients a an a_1 it is necessary to distinguish four cases.

Case I. $a \neq 0$, $a_1 \neq 0$. In this case the line OO_1 does not coincide with an one of the asymptotic tangents of the surfaces S, S_1 at the points O and O respectively. Eliminating α_1, β_1, γ_1, β_2, γ_3 from the system (8.5) and (8.4 we find that *there are two independent affine invariants determined by the neigh borhoods of the second order of the surfaces S, S_1 at the points O, O_1, namely,*

$$(8.6)\qquad J_{11}^* = \frac{a_1^2}{a^2} h^2(4ac - b^2), \qquad J_{12}^* = \frac{a^2}{a_1^2} h^2(4a_1c_1 - b_1^2).$$

Case II. $a = 0$, $a_1 \neq 0$. In this case the line OO_1 is an asymptotic tangen of the surface S at O, but it is not an asymptotic tangent of S_1 at O_1. Elimi nating α_1, β_1, γ_1, β_2, γ_3 from the system (8.5) and (8.4) we see that *in thi case the only affine invariant determined by the neighborhoods of the second order o the surfaces S, S_1 at the points O, O_1 is*

$$(8.7)\qquad J_{21}^* = (b_1^2 - 4a_1c_1)b^2h^4.$$

Case III. $a \neq 0$, $a_1 = 0$. The line OO_1 is an asymptotic tangent of S_1 a O_1 but is not an asymptotic tangent of S at O. Similarly as in the foregoin case the only affine invariant is found to be

$$(8.8) \qquad J^*_{31} = (b^2 - 4ac)b_1^2h^4.$$

Case IV. $a = 0$, $a_1 = 0$. The line OO_1 is an asymptotic tangent of both surfaces S, S_1 at the points O, O_1. Eliminating α_1, β_1, γ_1, β_2, γ_3 from the equations (8.5) and (8.4) we find that *in this case there is only one affine invariant determined by the neighborhoods of the second order of the surfaces S, S_1 at O, O_1, namely,*

$$(8.9) \qquad J^*_{41} = bb_1h^2.$$

9. **Metric and affine characterization of the invariants J^*_{ij}.** We consider separately the four cases mentioned in the foregoing number.

Case I. Let us call ω the angle between the tangent planes to S and S_1 at the points O and O_1 respectively, that is, the angle between the planes $y = 0$, $z = 0$ of our coordinate system to which are referred the equations (8.1) and (8.2) of S and S_1 respectively.

Let r be the radius of curvature at O of the plane curve in which the tangent plane to S_1 at O_1 (that is, the plane $y = 0$) intersects the surface S, and K the total curvature of S at O. Similarly, let r_1 be the radius of curvature at O_1 of the plane curve in which the tangent plane to S at O (that is, the plane $z = 0$) intersects the surface S_1, and K_1 the total curvature of S_1 at O_1. Then, if the plane $x = 0$ is chosen to be orthogonal to the x-axis, it can be readily demonstrated that

$$(9.1) \qquad \begin{aligned} 2r &= 1/a, & 2r_1 &= 1/a_1, \\ K &= (4ac - b^2)\sin^2\omega, & K_1 &= (4a_1c_1 - b_1^2)\sin^2\omega. \end{aligned}$$

From (9.1) and (8.6) we obtain the following metrical characterization of the affine invariants J^*_{11} and J^*_{12}:

$$(9.2) \qquad J^*_{11} = h^2\frac{r^2K}{r_1^2\sin^2\omega}, \qquad J^*_{12} = h^2\frac{r_1^2K_1}{r^2\sin^2\omega}.$$

From these formulas it follows in particular that the expression

$$(9.3) \qquad J^*_{11}J^*_{12} = h^4\frac{KK_1}{\sin^4\omega}$$

is an affine invariant. Buzano has proved that this invariant (9.3) is also a projective invariant [4]. Similarly

$$(9.4) \qquad J^*_{11}/J^*_{12} = (K/K_1)(r/r_1)^4$$

is an affine invariant. For the case $h = 0$ Hsiung has proved that this invariant (9.4) is also a projective invariant [7].

We now proceed to give an affine characterization of the affine invariants J^*_{11} and J^*_{12}. For this purpose let us consider the conjugate tangent t of the tangent OO_1 on the surface S and the conjugate tangent t_1 of the tangent O_1O on the surface S_1. Since we suppose that OO_1 is not an asymptotic tangent for S or S_1, t and t_1 are straight lines distinct from OO_1 whose equations are

$$(9.5)\qquad t: \quad y = -(2a/b)x, \qquad z = 0,$$

$$(9.6)\qquad t_1: \quad y = 0, \qquad z = -(2a_1/b_1)(x - h).$$

The one-parameter family of paraboloids which contain t and t_1 and are generated by lines which cut the line OO_1 and are parallel to the plane determined by the directions of t and t_1, is given by

$$(9.7)\qquad \left(x + \frac{b}{2a}y + \frac{b_1}{2a_1}z\right)(\lambda z + y) - \lambda hz = 0$$

where λ is the variable parameter.

The tangent lines at O of the curve in which the general paraboloid (9.7) intersects the surface S are given by

$$(9.8)\qquad \left(\frac{b}{2a} - \lambda hc\right)y^2 + (1 - \lambda hb)xy - \lambda hax^2 = 0, \qquad z = 0.$$

The necessary and sufficient condition that these two tangent lines coincide, that is to say, that the intersection curve of the paraboloid (9.7) and the surface S has a cusp at O, is

$$(9.9)\qquad 1 - \lambda^2h^2(4ac - b^2) = 0.$$

This equation gives two values of λ, say λ_1 and λ_2, each of which gives a paraboloid with the property that its intersection curve with the surface S has a cusp at O. Let us consider one of these paraboloids, for instance the paraboloid which corresponds to $\lambda = \lambda_1$. The tangent plane to this paraboloid at the point of infinity of the line OO_1 is the plane $y + \lambda_1 z = 0$ which intersects the surface S in the plane curve

$$(9.10)\qquad y + \lambda_1 z = 0, \qquad z = ax^2 - \lambda_1 bxz + \lambda_1^2 cz^2 + \cdots,$$

and the surface S_1 in the plane curve

$$(9.11)\qquad y + \lambda_1 z = 0, \qquad -\lambda z = a_1(x - h)^2 + b_1(x - h)z + c_1z^2 + \cdots$$

If r and r_1 are the radii of curvature of the plane curves (9.10) and (9.11) at the points O and O_1 respectively it is easily verified that

$$(9.12)\qquad \frac{r}{r_1} = -\frac{a_1}{a\lambda_1}.$$

Consequently, according to (9.9) and (8.6) we have

$$(9.13)\qquad \left(\frac{r}{r_1}\right)^2 = J_{11}^*.$$

Since the ratio r/r_1 according to §3 is an invariant with respect to affine transformations, the formula (9.13) gives an affine characterization of the invariant J^*_{11}. Evidently, by symmetry, the affine invariant J^*_{12} has an affine characterization entirely similar to the preceding one.*

Cases II, III, IV. In these cases, according to (9.1) where we put $a = 0$, $a_1 \neq 0$; $a \neq 0$, $a_1 = 0$ or $a = a_1 = 0$, the invariants J^*_{21}, J^*_{31} and J^*_{41} have the following metrical characterization:

$$J^*_{21} = J^*_{31} = J^{*2}_{41} = h^4 \frac{KK_1}{\sin^4 \omega},$$

that is, they coincide with the projective invariant given by Buzano [4]. Several projective characterizations of this invariant have been given by Bompiani [2].

10. **Summary.** In this paper we compute the simplest affine invariants of certain sets of curves and surfaces. They are:

(1) Two plane curves having a common tangent at two ordinary points. Two invariants are found, determined respectively by the neighborhoods of the second and the third order. These invariants are not projective invariants. The simplest projective invariants are determined by the neighborhoods of the fourth order (see [5]).

(2) Two intersecting plane curves at an ordinary point. Two invariants are found, determined by the neighborhoods of the third order. They are not projective. The simplest projective invariants are determined by the neighborhoods of the fourth order (see [6]).

(3) Two surfaces with common tangent plane at two ordinary points, with: (I) common tangent line in general position; three invariants are found J_{11}, J_{12}, J_{13} of which the combination $J_{13}J_{12}^{-1}J_{11}^{-2/3}$ is projective. (II, III) common tangent asymptotic on one surface; two invariants are found, not projective. (IV) common tangent asymptotic on both surfaces; one invariant is found, not projective. These invariants are determined by the neighborhoods of the second order. The mentioned projective invariant is the only one (see [8] and [12]).

(4) Two surfaces with common tangent line but distinct tangent planes at two ordinary points. Same division into cases as in (3). (I) Two invariants are found, J^*_{11}, J^*_{12} of which the product is projective; this projective invariant is the only one (see [4]). (II, III, IV) One invariant is found, which is projective. These invariants are determined by the neighborhoods of the second order. It is supposed that the distance h between the two points is $\neq 0$. For $h = 0$ there is in case (I) only one affine invariant J^*_{11}/J^*_{12}, which is also the only projective invariant (see [7]); in cases (II, III, IV) there are neither affine nor projective invariants determined by the neighborhoods of the second order.

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