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UNIFIED FIELD THEORIES OF EINSTEIN'S TYPE DEDUCED FROM A VARIATIONAL PRINCIPLE: CONSERVATION LAWS.

Dedicated to Professor A. Kawaguchi on the occasion of his 70th birthday.

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1. Introduction. In previous papers [4],[1] [5], [6] we have studied the following generalization of Einstein's unified theory of 1950. The space-time is assumed to be a four dimensional differentiable manifold endowed with a non-symmetric affine connection Γ^i_{jk} and a non-symmetric covariant tensor g_{ij}. The most general covariant tensor T_{ij}, which depends only on the connection Γ^i_{jk} and its first partial derivatives and it is at most of second degree as function of Γ^i_{jk}, is the tensor (2.13) where $\alpha, \beta, \ldots, \nu$ are arbitrary constants. Then we form the density $T_{ik}g^{ik}|g|^{1/2}$ and deduce the field equations from the corresponding variational principle. The field equations depend on the set of constants $\alpha, \beta, \ldots, \nu$. The classical Einstein's theory corresponds to $\alpha = 1$, $\beta = \gamma = \cdots = \nu = 0$. Particular cases have been considered by M. A. Tonnelat ([7], Note II) and ([8], p. 351–363) where related works of Nguyen Phong Chau (1963), J. Lévy (1959) and L. Bouche (1961) are mentioned. In [6] we have analyzed the conditions of the constants $\alpha, \beta, \ldots, \nu$ for the field equations to be invariant by λ-transformations or for T_{ij} to be a pseudo-hermitian tensor. In the present paper we shall give some complements of the general theory and, in particular, we establish the conservation laws or conservation identities which are satisfied for any set of variables (Γ^i_{jk}, g_{ij}) which satisfies the field equations.

Though we use small changes in the notations, the main references for the concepts and formulas in the sequel are the books of A. Lichnerowicz [3] and M. A. Tonnelat [7], [8].

2. Notations and field equations. Let Γ^i_{jk} be an affine connection and let

$$(2.1)\qquad \Delta^i_{jk} = \tfrac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}), \qquad S^i_{jk} = \tfrac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$$

be its symmetric and skewsymmetric parts (Δ^i_{jk} is a connection and S^i_{jk} is the tensor of torsion). Following Einstein we set

$$(2.2)\qquad S_i = S^k_{ik}\,.$$

Let g_{ij} be a non-symmetric tensor. If g denotes the determinant of g_{ij}, assumed $\neq 0$, we introduce the densities

$$(2.3)\qquad \mathfrak{G}_{ij} = g_{ij}|g|^{1/2}, \quad \mathfrak{H}_{ij} = \tfrac{1}{2}(\mathfrak{G}_{ij} + \mathfrak{G}_{ji}), \quad \mathfrak{F}_{ij} = \tfrac{1}{2}(\mathfrak{G}_{ij} - \mathfrak{G}_{ji})\,.$$

For any tensor or density of second rank, Einstein [1] introduced the mixed

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1) Numbers in brackets refer to the references at the end of the paper.

covariant derivative, obtained when one differentiates the first index with respect to Γ^i_{jk} and the second index with respect to $\Gamma^i_{jk} = \Gamma^i_{kj}$. We will denote this mixed covariant derivative by a vertical bar. For instance we put

$$(2.4) \qquad \mathfrak{G}^{ik}{}_{|s} = \mathfrak{G}^{ik}{}_{,s} + \Gamma^i_{ms}\mathfrak{G}^{mk} + \Gamma^k_{sm}\mathfrak{G}^{im} - \Delta^m_{sm}\mathfrak{G}^{ik},$$

where a comma denotes ordinary partial derivative. Denoting by a semi-colon the ordinary covariant derivative with respect to the connection Γ^i_{jk}, we have

$$(2.5) \qquad \mathfrak{G}^{ik}{}_{|s} = \mathfrak{G}^{ik}{}_{;s} + 2S^k_{sm}\mathfrak{G}^{im} - S_s\mathfrak{G}^{ik}.$$

Notice that for symmetric densities $\mathfrak{H}^{ik} = \mathfrak{H}^{ki}$ we have

$$(2.6) \qquad \mathfrak{H}^{ik}{}_{|s} = \mathfrak{H}^{ki}{}_{|s} + 2S^k_{sm}\mathfrak{H}^{im} + 2S^i_{ms}\mathfrak{H}^{mk}$$

and for skewsymmetric densities $\mathfrak{F}^{ik} = -\mathfrak{F}^{ki}$,

$$(2.7) \qquad \mathfrak{F}^{ik}{}_{|s} = -\mathfrak{F}^{ki}{}_{|s} + 2S^k_{sm}\mathfrak{F}^{im} + 2S^i_{ms}\mathfrak{F}^{mk}.$$

In particular, we have

$$(2.8) \qquad \mathfrak{H}^{ik}{}_{|i} = \mathfrak{H}^{ki}{}_{|i} + 2S_m\mathfrak{H}^{mk}$$

$$(2.9) \qquad \mathfrak{F}^{ik}{}_{|i} = -\mathfrak{F}^{ki}{}_{|i} + 2S^k_{im}\mathfrak{F}^{im} + 2S_m\mathfrak{F}^{mk}.$$

Notice also the formulas

$$(2.10) \qquad \mathfrak{F}^{ki}{}_{|i} = \mathfrak{F}^{ki}{}_{,i} + \Gamma^k_{mi}\mathfrak{F}^{mi} - S_i\mathfrak{F}^{ki},$$

$$(2.11) \qquad \mathfrak{H}^{ki}{}_{|i} = \mathfrak{H}^{ki}{}_{,i} + \Gamma^k_{mi}\mathfrak{H}^{mi} - S_i\mathfrak{H}^{ki}.$$

Though in general we shall use densities instead of tensors, we state the following formula:

$$(2.12) \qquad \mathfrak{G}^{ik}{}_{|s} = |g|^{1/2}g^{ik}{}_{|s} + g^{ik}(|g|^{1/2})_{,s} - g^{ik}|g|^{1/2}\Delta^t_{st},$$

which holds good for any covariant derivative. By means of (2.12) the field equations in the sequel may be expressed in terms of the tensor g_{ij}.

The most general covariant tensor T_{ik}, which depends only on the connection Γ^i_{jk} and its first partial derivatives and it is at most of second degree as functions of Γ^i_{jk}, is

$$(2.13) \qquad \begin{aligned} T_{ik} = {} & \alpha R_{ik} + \beta(\Delta^m_{im,k} - \Delta^m_{km,i}) + \gamma S^m_{ik;m} + \delta S^q_{ir}S^r_{kq} \\ & + \epsilon S_{i;k} + \phi S_{k;i} + \mu S_m S^m_{ik} + \nu S_i S_k, \end{aligned}$$

where R_{ik} is the Ricci tensor

$$(2.14) \qquad R_{ik} = \Gamma^m_{ik,m} - \Gamma^m_{im,k} + \Gamma^m_{jm}\Gamma^j_{ik} - \Gamma^m_{jk}\Gamma^j_{im}.$$

For a proof, see [6].

The variational principle

$$(2.15) \qquad \delta\int T_{ik}\mathfrak{G}^{ik}\,d\tau = 0$$

(where $d\tau = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$) gives rise to the field equations

$$(2.16) \qquad \mathfrak{R}^{rs}_{r} = 0, \qquad T_{ik} = 0$$

(see [6], where equations (19) are clearly misprinted), where

$$\begin{aligned}(2.17)\quad \mathfrak{R}^{qs}_{r} &= \alpha(-\mathfrak{G}^{qs}{}_{|r} + \mathfrak{G}^{qs}S_r + (\mathfrak{G}^{qt}{}_{|t} + \mathfrak{G}^{qt}S_t)\delta^s_r) - \beta(\mathfrak{F}^{qt}{}_{,t}\delta^s_r + \mathfrak{F}^{st}{}_{,t}\delta^q_r)\\ &+ \gamma(-\mathfrak{F}^{qs}{}_{|r} + \mathfrak{F}^{ik}S^q_{ik}\delta^s_r + \mathfrak{F}^{qs}S_r) + \delta(\mathfrak{H}^{qt}S^s_{tr} - \mathfrak{H}^{st}S^q_{tr})\\ &+ \varepsilon\{-\tfrac{1}{2}(\mathfrak{G}^{qt}{}_{,t} + \mathfrak{G}^{ik}\Gamma^q_{ik})\delta^s_r + \tfrac{1}{2}(\mathfrak{G}^{st}{}_{,t} + \mathfrak{G}^{ik}\Gamma^s_{ik})\delta^q_r - \mathfrak{G}^{qs}S_r\}\\ &+ \phi\{\tfrac{1}{2}(-\mathfrak{G}^{tq}{}_{,t} - \mathfrak{G}^{ik}\Gamma^q_{ki})\delta^s_r + \tfrac{1}{2}(\mathfrak{G}^{ts}{}_{,t} + \mathfrak{G}^{ik}\Gamma^s_{ki})\delta^q_r - \mathfrak{G}^{sq}S_r\}\\ &+ \mu(\tfrac{1}{2}\mathfrak{F}^{ik}S^q_{ik}\delta^s_r - \tfrac{1}{2}\mathfrak{F}^{ik}S^s_{ik}\delta^q_r + \mathfrak{F}^{qs}S_r) + \nu(\mathfrak{H}^{qt}S_t\delta^s_r - \mathfrak{H}^{st}S_t\delta^q_r)\,.\end{aligned}$$

By means of (2.12) it is easy to express these equations in terms of the tensor g^{qs} instead of the density $\mathfrak{G}^{qs}$. For instance, for the classical case $\alpha = 1$, $\beta = \gamma = \cdots = \nu = 0$, we get

$$\begin{aligned}\mathfrak{G}^{qs}{}_{|r} - \mathfrak{G}^{qs}S_r - (\mathfrak{G}^{qt}{}_{|t} + \mathfrak{G}^{qt}S_t)\delta^s_r &= |g|^{1/2}(g^{qs}{}_{;r} - (\Gamma^t_{rt} - \gamma_r)g^{qs}\\ &+ 2g^{qt}S^s_{rt} - \delta^s_r(g^{qt}{}_{;t} - (\Gamma^i_{ti} - \gamma_t)g^{qt}) = 0\,,\end{aligned}$$

where $\gamma_t = (|g|^{1/2})_{,t}$, in accordance with Lichnerowicz [3].

The expression (2.17) takes a simple form if the constants $\alpha, \beta, \gamma, \ldots, \nu$ are such that

$$(2.18)\qquad \alpha \neq 0\,,\quad \alpha + \gamma \neq 0\,,\quad \delta = 0\,,\quad (2\phi + \mu)\alpha + (\varepsilon + \phi)\gamma = 0\,,$$

a set of conditions which we will assume satisfied from now on.

In this case it is useful to introduce the new connection

$$(2.19)\qquad L^q_{tr} = \Gamma^q_{tr} + \tfrac{1}{3}(2 - (\varepsilon + \phi)/\alpha)\delta^q_t S_r - \tfrac{1}{3}((\varepsilon + \phi)/\alpha)\delta^q_r S_t\,,$$

which is such that

$$(2.20)\qquad L_t = \tfrac{1}{2}(L^q_{tq} - L^q_{qt}) = 0\,.$$

Then the first equations (2.16) split into (see [6])

$$\begin{aligned}(2.21)\qquad \mathfrak{Q}^{qs}_{r} &\equiv -\alpha\mathfrak{G}^{qs}{}_{|r}(L) - \gamma\mathfrak{F}^{qs}{}_{|r}(L) + \tfrac{1}{3}(\gamma + 2\beta)\delta^s_r\mathfrak{F}^{qt}{}_{,t}\\ &+ \tfrac{1}{3}(-2\alpha + 2\beta - \gamma)\delta^q_r\mathfrak{F}^{st}{}_{,t} = 0\,,\end{aligned}$$

$$\begin{aligned}(2.22)\quad \mathfrak{R}^{is}_{i} &\equiv \tfrac{3}{2}(\varepsilon - \phi - \mu)\mathfrak{F}^{st}{}_{|t}(L) + (2\alpha - 5\beta + \gamma + \tfrac{3}{2}\mu)\mathfrak{F}^{st}{}_{,t} + \tfrac{3}{2}(\varepsilon + \phi)\mathfrak{H}^{st}{}_{|t}(L)\\ &- \{(\varepsilon + \phi)(2\alpha - \varepsilon - \phi)/\alpha + \delta + 3\nu\}S_t\mathfrak{H}^{st} = 0\,,\end{aligned}$$

where the mixed covariant derivatives refer to the connection L^q_{tr}.

From (2.21) having (2.8) and (2.9) into account, we deduce

$$(2.23)\qquad \mathfrak{Q}^{is}_{i} - \mathfrak{Q}^{si}_{i} \equiv 2(\alpha + \gamma)(\mathfrak{F}^{st}{}_{|t}(L) - L^s_{tm}\mathfrak{F}^{tm} - \mathfrak{F}^{st}{}_{,t}) = 0\,.$$

Hence, according to (2.10) and (2.20), the equations $\mathfrak{Q}^{is}_{i} - \mathfrak{Q}^{si}_{i} = 0$ are identically satisfied and that justifies the addition of (2.22) as field equations. On the other side (2.21) gives

$$(2.24)\quad \mathfrak{Q}^{is}_{i} + \mathfrak{Q}^{si}_{i} \equiv -2(\alpha + \gamma)L^s_{tm}\mathfrak{F}^{tm} - 2\alpha\mathfrak{H}^{st}{}_{|t}(L) + \tfrac{4}{3}(2\beta - \alpha)\mathfrak{F}^{st}{}_{,t} = 0\,.$$

Hence, from (2.23) and (2.24), assuming $\alpha + \gamma \neq 0$, we have

$$(2.25)\qquad \mathfrak{F}^{st}{}_{|t}(L) = L^s_{tm}\mathfrak{F}^{tm} + \mathfrak{F}^{st}{}_{,t}\,,$$

$$(2.26)\qquad \alpha\mathfrak{H}^{st}{}_{|t}(L) = \tfrac{2}{3}(2\beta - \alpha)\mathfrak{F}^{st}{}_{,t} - (\alpha + \gamma)L^s_{tm}\mathfrak{F}^{tm}\,.$$

Substituting (2.25) and (2.26) into (2.22) and having (2.18) into account, we get the interesting relation

$$A\mathfrak{F}^{si}{}_{,i} + BS_i\mathfrak{H}^{si} = 0\,, \tag{2. 27}$$

where

$$A = 2\alpha - 5\beta + 5\beta(\varepsilon + \phi)/\alpha - \varepsilon - 4\phi + \gamma\,, \tag{2. 28}$$

$$B = -(\varepsilon + \phi)(2\alpha - \varepsilon - \phi)/\alpha - \delta - 3\nu\,. \tag{2. 29}$$

Notice that by means of the connection L^i_{kr} the tensor T_{ik} writes

$$T_{ik} = T_{ik}(L) + \tfrac{1}{3}A(S_{i,k} - S_{k,i}) - \tfrac{1}{3}BS_iS_k\,, \tag{2. 30}$$

and the second set of field equations (2.16) writes

$$T_{ik}(L) + \tfrac{1}{3}A(S_{i,k} - S_{k,i}) - \tfrac{1}{3}BS_iS_k = 0\,. \tag{2. 31}$$

The field equations are (2.21), (2.22) and (2.31) from which the relations (2.25), (2.26) and (2.27) follow. The unknowns are L^i_{kr}, $\mathfrak{G}^{ik}$, S_i. The components Γ^i_{kr} are then given by (2.19) from which the relation $\Gamma^q_{iq} - \Gamma^q_{qi} = 2S_i$ follows.

3. **Particular cases of the field equations.** Taking account of (2.27) we see that there are two important particular cases to be considered:

a) $B = 0$, $A \neq 0$. Then (2.27) gives

$$\mathfrak{F}^{si}{}_{,i} = 0\,, \tag{3. 1}$$

and equations (2.21) give

$$\alpha\mathfrak{G}^{sr}{}_{|r}(L) + \gamma\mathfrak{F}^{sr}{}_{|r}(L) = 0\,. \tag{3. 2}$$

Equations (2.31), if $B = 0$, may be written as

$$T_{(ik)}(L) = 0\,, \qquad T_{[ik],j}(L) + T_{[kj],i}(L) + T_{[ji],k}(L) = 0\,, \tag{3. 3}$$

where ( ) denotes the symmetric part and [ ] the skewsymmetric part of the tensor $T_{ik}(L)$. The field equations are (3.1), (3.2) and (3.3). When $\gamma = 0$, this system reduces to the so called "weak system" of Einstein. Thus, we have proved that any tensor T_{ik} (2.13) such that the constants $\alpha, \beta, \gamma, \ldots, \nu$ satisfy the conditions (2.18) and $B = 0$, $\gamma = 0$ (B given by (2.29)) gives rise to the field equations of the weak system of Einstein.

b) $A = 0$, $B \neq 0$. Assuming that the determinant $\|\mathfrak{H}^{si}\| \neq 0$, (2.27) gives

$$S_i = 0\,. \tag{3. 4}$$

Then, according to (2.19) we have $L^i_{kr} = \Gamma^i_{kr}$. Having into account (2.18), (2.25), (2.26) and (3.4), it follows that equations (2.22) are identically satisfied, so that the field equations reduce to (2.21), (2.31) and (3.4) which may be written as

$$\begin{gathered}\alpha\mathfrak{G}^{sr}{}_{|r} + \gamma\mathfrak{F}^{sr}{}_{|r} - \tfrac{1}{3}(\gamma + 2\beta)\delta^s_r\mathfrak{F}^{ri}{}_{,i} - \tfrac{1}{3}(-2\alpha + 2\beta - \gamma)\delta^s_r\mathfrak{F}^{ri}{}_{,i} = 0\,,\\ S_i = 0\,, \qquad T_{ik} = 0\,.\end{gathered} \tag{3. 5}$$

The more simple case corresponds to $\gamma = 0$, $\beta = 0$. Then the system (3.5) takes the simple form

$$\mathfrak{G}^{sr}{}_{|r} = -\tfrac{2}{3}\delta^s_r\mathfrak{F}^{ri}{}_{,i}\,, \qquad S_i = 0\,, \quad T_{ik} = 0\,, \tag{3. 6}$$

where

$$T_{ih} = \alpha R_{ih} + (2\alpha - 4\phi) S_{i;h} + \phi S_{h;i} - 2\phi S_m S^m_{ih} + \nu S_i S_h \,.$$

These field equations are valid for any set of constants $\alpha \neq 0, \phi, \nu$. We have applied that in the present case we have $\varepsilon = 2\alpha - 4\phi$, $\mu = -2\phi$. For instance, the particular tensors ${}^1R_{ih}$, ${}^2R_{ih}$ considered by Tonnelat ([7], pp. 129–130) belong to this class. Indeed, ${}^1R_{ih}$ corresponds to $\alpha = 1$, $\phi = \frac{2}{3}$, $\nu = 0$ and ${}^2R_{ih}$ corresponds to $\alpha = 1$, $\phi = \frac{1}{3}$, $\nu = -\frac{1}{3}$.

The tensor ${}^3R_{ih}$ of Tonnelat ([7], pp. 129–130) corresponds to $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = 0$, $\delta = 0$, $\varepsilon = \frac{2}{3}$, $\phi = \frac{1}{3}$, $\mu = -\frac{2}{3}$, $\nu = -\frac{1}{3}$ and hence $A = 0$, $B = 0$. Equations (2.27) are identically satisfied and the field equations reduce to (2.21) and (2.31) which in this case may be written as

$$(3.\ 7) \qquad -\mathfrak{G}^{qs}{}_{|r}(L) + \tfrac{1}{3}\delta^s_r \mathfrak{F}^{qi}{}_{,i} - \tfrac{1}{3}\delta^q_r \mathfrak{F}^{si}{}_{,i} = 0 \,, \qquad {}^3R_{ih}(L) = 0 \,.$$

c) As a last example we consider the Einstein tensor (see [1])

$$(3.\ 8) \qquad E_{ih} = -\tfrac{1}{2}(\Delta^m_{im,h} + \Delta^m_{hm,i}) + \Gamma^m_{ih,m} + \Gamma^s_{ih}\Delta^m_{sm} - \Gamma^m_{is}\Gamma^s_{mh} \,,$$

which corresponds to

$$(3.\ 9) \qquad \alpha = 1\,, \quad \beta = \tfrac{1}{2}\,, \quad \varepsilon = 1\,, \quad \gamma = \delta = \phi = \mu = \nu = 0 \,.$$

We have $A = 1$, $B = -1$. The connection (2.19) writes

$$(3.\ 10) \qquad L^q_{ir} = \Gamma^q_{ir} + \tfrac{1}{3}\delta^q_i S_r - \tfrac{1}{3}\delta^q_r S_i \,,$$

and the field equations (2.21) write

$$(3.\ 11) \qquad -\mathfrak{G}^{qs}{}_{|r}(L) + \tfrac{1}{3}\delta^s_r \mathfrak{F}^{qi}{}_{,i} - \tfrac{1}{3}\delta^q_r \mathfrak{F}^{si}{}_{,i} = 0 \,.$$

The field equations (2.22) are equivalent to (2.27), i.e.

$$(3.\ 12) \qquad \mathfrak{F}^{si}{}_{,i} = S_i \mathfrak{H}^{si} \,.$$

The field equations are thus (3.11), (3.12) and $E_{ih} = 0$. Adding the condition $S_i = 0$ we get the "strong system" of Einstein $\mathfrak{G}^{qs}{}_{|r} = 0$, $S_i = 0$, $E_{ih} = 0$ which, however, is not deducible from a variational principle.

4. Conservation laws. To get the identities of conservation we will follow a similar approach to that of Lichnerowicz and Weyl for the case $\alpha = 1$, $\beta = \gamma = \ldots = \nu = 0$ (see Lichnerowicz [2]).

Let C be a domain of the space-time of boundary ∂C and let ξ^i be a vector field which vanishes on ∂C. Consider

$$(4.\ 1) \qquad I = \int_C \mathfrak{G}^{ih} T_{ih}\, d\tau = \int_C \mathfrak{T}\, d\tau \,,$$

where $\mathfrak{T} = T_{ih} g^{ih} |g|^{1/2} = g^{ih}\mathfrak{T}_{ih}$ and $d\tau = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

The Lie derivative with respect to the field ξ^i is

$$(4.\ 2) \qquad L_\xi I = \int_C L_\xi \mathfrak{T}\, d\tau = \int_C (\mathfrak{T}\xi^i)_{,i}\, d\tau \,.$$

By means of the Stokes' theorem this integral transforms into an integral extended on ∂C, which is zero since ξ^i vanishes on ∂C. Thus we have

$$(4.\ 3) \qquad L_\xi I = 0 \,.$$

On the other side, recall that the Lie derivative of Γ^i_{jk} is a mixed tensor of contravariant valence 1 and covariant valence 2, given by

$$(4.4)\qquad L_\xi \Gamma^i_{jk} = \xi^i{}_{,jk} + \xi^m \Gamma^i_{jk,m} - \xi^i{}_{,m}\Gamma^m_{jk} + \xi^m{}_{,j}\Gamma^i_{mk} + \xi^m{}_{,k}\Gamma^i_{jm}$$

(see Yano [9]). Thus, assuming that the vector field ξ^i and its first and second derivatives vanish on ∂C, we have, on ∂C, $L_\xi \Gamma^i_{jk} = 0$. According to the variational principle from which the field equations are deduced, this condition implies that

$$\int_C (L_\xi T_{ih})\mathfrak{G}^{ih}\, d\tau = 0\,,$$

and (4.3) gives

$$(4.5)\qquad L_\xi I = \int_C T_{ih} L_\xi \mathfrak{G}^{ih}\, d\tau = 0\,.$$

As is well known (see Yano [9]) we have

$$(4.6)\qquad L_\xi \mathfrak{G}^{ih} = \xi^m \mathfrak{G}^{ih}{}_{,m} + \mathfrak{G}^{ih}\xi^m{}_{,m} - \mathfrak{G}^{mh}\xi^i{}_{,m} - \mathfrak{G}^{im}\xi^h{}_{,m}\,,$$

and hence

$$\begin{aligned}(4.7)\quad T_{ih}L_\xi \mathfrak{G}^{ih} &= \xi^m T_{ih}\mathfrak{G}^{ih}{}_{,m} + T_{ih}\mathfrak{G}^{ih}\xi^m{}_{,m} - (T_{sh}\mathfrak{G}^{mh} + T_{is}\mathfrak{G}^{im})\xi^s{}_{,m}\\ &= [T_{ih}\mathfrak{G}^{ih}\xi^m - (T_{sh}\mathfrak{G}^{mh} + T_{is}\mathfrak{G}^{im})\xi^s]_{,m} - \mathfrak{G}^{ih}\xi^m T_{ih,m}\\ &\quad + (T_{sh}\mathfrak{G}^{mh} + T_{is}\mathfrak{G}^{im})_{,m}\xi^s\,.\end{aligned}$$

Substituting this expression in (4.5), applying then the Stokes' theorem and having into account that the vector field ξ^i is an arbitrary vector field which vanishes on ∂C (together with its first and second derivatives), we get

$$(4.8)\qquad (T_{sh}\mathfrak{G}^{mh} + T_{is}\mathfrak{G}^{im})_{,m} - \mathfrak{G}^{ih}T_{ih,s} = 0\,.$$

Putting

$$(4.9)\qquad \mathfrak{W}^m_s = \tfrac{1}{2}(T_{sh}\mathfrak{G}^{mh} + T_{is}\mathfrak{G}^{im}) - \tfrac{1}{2}\delta^m_s \mathfrak{G}^{ih}T_{ih},$$

(4.8) may be written as

$$(4.10)\qquad \mathfrak{W}^m_{s,m} + \tfrac{1}{2}T_{ih}\mathfrak{G}^{ih}{}_{,s} = 0\,,$$

which is the first form of the four identities of conservation.

These identities refer to the connection Γ^l_{ir}. If we want to introduce the connection L^l_{ir} (2.19) which gives rise to the field equations (2.21) and (2.22), notice that substituting the expression (2.30) of T_{ih} into (4.9) we get

$$\begin{aligned}(4.11)\ \mathfrak{W}^m_s &= \tfrac{1}{2}(T_{sh}(L) + \tfrac{1}{3}A(S_{s,h} - S_{h,s}) - \tfrac{1}{3}BS_sS_h)\mathfrak{G}^{mh} + \tfrac{1}{2}(T_{hs}(L) + \tfrac{1}{3}A(S_{h,s} - S_{s,h})\\ &\quad - \tfrac{1}{3}BS_sS_h)\mathfrak{G}^{hm} - \tfrac{1}{2}\delta^m_s\mathfrak{G}^{ij}(T_{ij}(L) + \tfrac{1}{3}A(S_{i,j} - S_{j,i}) - \tfrac{1}{3}BS_iS_j)\\ &= \tfrac{1}{2}T_{sh}(L)\mathfrak{G}^{mh} + \tfrac{1}{2}T_{hs}(L)\mathfrak{G}^{hm} - \tfrac{1}{2}\delta^m_s\mathfrak{G}^{ij}T_{ij}(L) + \tfrac{1}{3}A(S_{s,h} - S_{h,s})\mathfrak{F}^{mh}\\ &\quad - \tfrac{1}{6}\delta^m_s A(S_{i,j} - S_{j,i})\mathfrak{F}^{ij} - \tfrac{1}{3}BS_sS_h\mathfrak{H}^{mh} + \tfrac{1}{6}B\delta^m_s\mathfrak{H}^{ij}S_iS_j\,.\end{aligned}$$

Putting

$$(4.12)\quad \mathfrak{N}^m_s = \tfrac{1}{2}T_{sh}(L)\mathfrak{G}^{mh} + \tfrac{1}{2}T_{hs}(L)\mathfrak{G}^{hm} - \tfrac{1}{2}\delta^m_s\mathfrak{G}^{ij}T_{ij}(L) - \tfrac{1}{3}BS_sS_h\mathfrak{H}^{mh}\,,$$

and having into account the value of $\mathfrak{W}^m_s$, an easy calculation shows that (4.10) may be written as

$$(4.\ 13)\quad \mathfrak{N}^{m}_{s,m} + \tfrac{1}{2}T_{ij}(L)\mathfrak{G}^{ij}{}_{,s} + \tfrac{1}{3}S_{h,s}(A\mathfrak{F}^{hi}{}_{,i} + BS_i\mathfrak{H}^{hi}) - \tfrac{1}{3}AS_{s,h}\mathfrak{F}^{hi}{}_{,i} = 0\,,$$

which is the second form of the identities of conservation.

Notice the relation

$$(4.\ 14)\quad \mathfrak{N}^{m}_{s} = \mathfrak{W}^{m}_{s} - \tfrac{1}{3}A(S_{s,h} - S_{h,s})\mathfrak{F}^{mh} + \tfrac{1}{6}\delta^{m}_{s}A(S_{i,j} - S_{j,i})\mathfrak{F}^{ij} - \tfrac{1}{6}\delta^{m}_{s}BS_iS_j\mathfrak{H}^{ij}\,.$$

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