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GEODESICS IN GÖDEL-SYNGE SPACES.

Dedicated to Professor Akitsugu Kawaguchi on the occasion of his 80th birthday.

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§1. **Introduction.** J. L. Synge [8][1] has considered 4-dimensional spaces which element of line has the form

$$ds^2 = dx_1^2 + 2h(x_4)dx_1dx_2 + g(x_4)dx_2^2 - dx_3^2 - dx_4^2, \tag{1.1}$$

where h and g are functions of x_4 only, which we assume of class C^∞ and g always positive. For $2g = h^2$ and $h = \exp x_4$, (1.1) reduces to the Gödel's classical line element ([5],[6]).

According to a result of Bampi-Zordan [2] (see also J. P. Wright [9]), if the metric (1.1) satisfies the Einstein equations of General Relativity for a perfect fluid, then all possible solutions for h and g give rise to spaces which are isometric to each other and therefore isometric to Gödel's space. However, the spaces with the metric (1.1) have some particular properties whose relations with the Gödel's case may have some interest. Our purpose is to study the geodesic lines of (1.1) and to compare with the work of S. Chandrasekhar and J. P. Wright on geodesics in Gödel's universe [3]. We will give, first, an isometric embedding of (1.1) in a pseudoeuclidean space of 10 dimensions, from which some properties on closed time-like curves can be deduced. Finally, we give an isometric embedding in a 10-dimensional pseudoeuclidean space of Gödel's space in cylindrical coordinates and deduce some consequences.

§2. **Isometric embedding of the arc element of Gödel-Synge in a pseudoeuclidean space of dimension 10.** The line element (1.1) is isometrically embedded in the 10-dimensional pseudoeuclidean space

$$ds^2 = \sum_1^5 dz_i^2 - \sum_6^{10} dz_i^2 \tag{2.1}$$

by the functions

$$\begin{aligned}
&z_1 = x_1, \quad z_2 = \sqrt{g}\cos x_2, \quad z_3 = \sqrt{g}\sin x_2, \\
&z_4 = \sqrt{2h}\cos\tfrac{1}{2}(x_1+x_2), \quad z_5 = \sqrt{2h}\sin\tfrac{1}{2}(x_1+x_2), \quad z_6 = x_3, \quad z_7 = x_4, \\
&z_8 = \sqrt{g}, \quad z_9 = \sqrt{2h}\cos\tfrac{1}{2}(x_1-x_2), \quad z_{10} = \sqrt{2h}\sin\tfrac{1}{2}(x_1-x_2).
\end{aligned} \tag{2.2}$$

From this embedding it follows that the x_2-curves are closed curves and, according to (1.1) they are time-like curves ($ds^2>0$). Thus, *the Gödel-Synge spaces are covering spaces of a non causal space.*

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1) Numbers in brackets refer to the references at the end of the paper.

Using equations (3.1) of the next section, it follows that these x_2-curves are geodesics only if $g'=0$, a case of little interest. On isometric embeddings of relativisitic Riemann spaces in pseudoeuclidean spaces, see J. Rosen [7].

§3. **Equations of geodesics.** The equations of the geodesic curves of the Gödel-Synge space defined by the line element (1.1) are

$$\begin{aligned} &\Delta\dot{u}_1+hh'u_1u_4+(hg'-gh')u_2u_4=0\,,\\ &\Delta\dot{u}_2+(hh'-g')u_2u_4-h'u_1u_4=0\,,\\ &\dot{u}_3=0\,,\\ &\dot{u}_4+h'u_1u_2+\tfrac{1}{2}g'u_2u_2=0 \end{aligned} \tag{3.1}$$

with the relation

$$u_1^2+2hu_1u_2+gu_2^2-u_3^2-u_4^2=1\,, \tag{3.2}$$

where

$$\Delta=h^2-g\,,\qquad u_i=dx_i/ds\,,\qquad h'=dh/dx_4\,,\qquad g'=dg/dx_4\,. \tag{3.3}$$

From (3.1) it follows that the x_1-curves, x_3-curves and x_4-curves are geodesics. If $x_4=$const. and h, g are assumed not constants, the geodesics are $x_1=a_1s+b_1$, $x_2=$const., $x_3=a_3s+b_3$, $x_4=$ const.

Assume that x_4 is not constant, and therefore $u_4\neq 0$. From (3.1) we have

$$\dot{u}_4u_4+\Delta u_2\dot{u}_2+(hh'-\tfrac{1}{2}g')u_2^2u_4=0 \tag{3.4}$$

and therefore

$$u_4^2+\Delta u_2^2=B^2=\text{const}\,. \tag{3.5}$$

From the third equation of (3.1) we deduce $x_3=Cs+c_3$ and then, (3.2) and (3.5) give

$$(u_1+hu_2)^2=1+B^2+C^2=\tfrac{1}{2}D^2\,, \tag{3.6}$$

where we have introduced the constant D in order to follow the notation of Chandrasekhar-Wright in [3].

From (3.6) and (3.5) we have

$$u_1=D/\sqrt{2}-hu_2=D/\sqrt{2}-h\{(B^2-u_4^2)/\Delta\}^{1/2} \tag{3.7}$$

and from (3.5) and the last equation of (3.1) we deduce

$$\dot{u}_4+h'(D/\sqrt{2})\{(B^2-u_4^2)/\Delta\}^{1/2}+(\tfrac{1}{2}g'-hh')(B^2-u_4^2)/\Delta=0\,. \tag{3.8}$$

With the change of variable

$$u_4=B\sin\theta\,, \tag{3.9}$$

we get

$$\dot{\theta}=\{(hh'-g'/2)/\Delta\}B\cos\theta-h'D/\sqrt{2\Delta}\,. \tag{3.10}$$

This equation suggests the study of the particular case

$$2g=h^2 \tag{3.11}$$

for which (3.10) reduces to

$$d\theta = \frac{B\cos\theta - D}{B\sin\theta}\frac{dh}{h}, \tag{3.12}$$

and therefore

$$D - B\cos\theta = K/h, \tag{3.13}$$

where K is a new integration constant. From (3.9) it follows that

$$u_4 = [B^2 - (D - K/h)^2]^{1/2}, \quad \int_0^{x_4} [B^2 - (D - K/h)^2]^{-1/2} dx_4 = s - s_0, \tag{3.14}$$

from which we deduce the function $x_4 = x_4(s)$.

From (3.7), (3.11) and (3.13) we deduce

$$u_1 = (1/\sqrt{2})(2K/h - D) \tag{3.15}$$

and using (3.14) we have

$$x_1 = (1/\sqrt{2}) \int_0^{x_4} (2K/h - D)(B^2 - (D - K/h)^2)^{-1/2} dx_4. \tag{3.16}$$

Using (3.5) and since $\Delta = h^2 - g = h^2/2$, we have

$$u_2 = \sqrt{2}(D - K/h)/h \tag{3.17}$$

and therefore

$$x_2 = \sqrt{2} \int_0^{x_4} \frac{(D - K/h)}{h[B^2 - (D - K/h)^2]^{1/2}} dx_4. \tag{3.18}$$

Finally, from the third equation of (3.1) and (3.14) we have

$$x_3 = C \int_0^{x_4} \frac{dx_4}{[B^2 - (D - K/h)^2]^{1/2}}. \tag{3.19}$$

Equations (3.16), (3.18), (3.19) and (3.14) are the explicit equations of the geoedesics of the Gödel-Synge space (1.1) in the case $2g = h^2$. Notice that the null geodesics correspond to the case in which the constants B, C, D are related by the condition

$$D^2 - 2B^2 - 2C^2 = 0 \tag{3.20}$$

with the relation $D^2 > B^2$ which is a consequence of (3.6).

For the particular case of Gödel's universe ($h = \exp x_4$) these null geodesics have been considered by Abdel Megied and M. Dautcourt [1].

§ 4. The geodesics of Gödel's universe. We want to show that when $h = \exp x_4$ (Gödel's case) the preceding formulae coincide with those given by Chandrasekhar and Wright in [3].

Putting

$$y = 1/h = \exp(-x_4), \qquad dx_4 = -dy/y \tag{4.1}$$

and assuming that $x_4 = 0$ corresponds to $s = 0$, from (3.14) we have

(4.2) $$s=-\int_{1}^{y}\frac{dy}{y(-K^{2}y^{2}+2DKy+B^{2}-D^{2})^{1/2}}.$$

Since $B^2-D^2<0$ and $-4K^2B^2<0$ we have

(4.3) $$s=+(D^{2}-B^{2})^{-1/2}\left[-\arc\sin\frac{DKy+B^{2}-D^{2}}{KBy}+\arc\sin\frac{DK+B^{2}-D^{2}}{KB}\right].$$

Introducing the parameter σ defined by

(4.4) $$s-(D^{2}-B^{2})^{-1/2}\ \arc\sin\{(DK+B^{2}-D^{2})/KB\}=(D^{2}-B^{2})^{-1/2}(2\sigma+\pi/2),$$

we get

(4.5) $$\cos 2\sigma=-(DKy+B^{2}-D^{2})/KBy,$$

and therefore

(4.6) $$y=\frac{D^{2}-B^{2}}{K(D+B\cos 2\sigma)}=\frac{D-B}{K}\,\frac{1+\tan^{2}\sigma}{1+\alpha\tan^{2}\sigma},$$

where

(4.7) $$\alpha=(D-B)/(D+B).$$

Applying (4.1) we have

(4.8) $$x_{4}=\log\frac{1+\alpha\tan^{2}\sigma}{1+\tan^{2}\sigma},\qquad s=2\sigma/(D^{2}-B^{2})^{1/2},$$

where we have normalized by the condition that to $\sigma=0$ corresponds $x_4=0$ (which implies that $K=D-B$).

From (3.16) and (4.1) we deduce

(4.9) $$dx_{1}=\frac{-\sqrt{2}\,Kdy}{(-K^{2}y^{2}+2DKy+B^{2}-D^{2})^{1/2}}-\frac{D}{\sqrt{2}}ds,$$

and therefore

(4.10) $$x_{1}=-(D/\sqrt{2})s+\sqrt{2}\arc\sin\{(D-Ky)/B\}-\sqrt{2}\ \arc\sin\{(D-K)/B\}$$

and assuming that for $\sigma=0$ we have $x_1=0$, we get

(4.11) $$x_{1}=-\{\sqrt{2}\,D/(D^{2}-B^{2})^{1/2}\}\sigma+\sqrt{2}\ \arc\tan\{(1-\alpha\tan^{2}\sigma)/(2\sqrt{\alpha}\ \tan\alpha)\}-\pi/\sqrt{2},$$

which can be written

(4.12) $$x_{1}=-\{\sqrt{2}\,D/(D^{2}-B^{2})^{1/2}\}\sigma-2\sqrt{2}\ \arc\tan(\sqrt{\alpha}\ \tan\sigma).$$

In order to calculate x_2 we have

(4.13) $$dx_{2}=-\sqrt{2}\ (D-Ky)(B^{2}-(D-Ky)^{2})^{-1/2}dy,$$

and therefore

(4.14) $$x_{2}=-(\sqrt{2}/K)(B^{2}-(D-Ky)^{2})^{1/2}+\text{constant},$$

or, by (3.14) $x_2=-(\sqrt{2}/K)u_4+\text{const.}$ and using (4.8) we get

(4.15) $$x_2 = -(\sqrt{2}/K)\{(\alpha-1)\tan\sigma/(1+\alpha\tan^2\sigma)\}(D^2-B^2)^{1/2}+\text{const}\,.$$

Under the assumption that $K=D-B$, (4.15) can be written

(4.16) $$x_2=(D^2-B^2)^{-1/2}2\sqrt{2}\,B\tan\sigma/(1+\alpha\tan^2\sigma)+\text{const}\,.$$

Finally, for x_3, according to (3.19) and (4.4) we have

(4.17) $$x_3=2C(D^2-B^2)^{-1/2}\sigma+\text{const}\,.$$

(4.8), (4.12), (4.16) and (4.17) are the equations of the geodesics in Gödel's space according to Chandrasekhar-Wright [3].

§5. Gödel's universe in cylindrical coordinates. Gödel ([5], [6]) has given a new form of his line element by transforming to a system of cylindrical coordinates r, ϕ, t, z defined by the following formulas of transformation:

(5.1) $$\begin{gathered}\exp x_4=\cosh r+\sinh 2r\cos\phi\,,\\ x_2\exp x_4=\sqrt{2}\sinh 2r\sin\phi\,,\\ \phi+(1/\sqrt{2})(x_1-2t)=2\arctan(\exp(-2r)\tan(\phi/2))\,,\\ x_3=2z\,.\end{gathered}$$

The line element (1.1) becomes

(5.2) $$ds^2=4(dt^2-dr^2-dz^2+(\sinh^4 r-\sinh^2 r)d\phi^2+2\sqrt{2}\sinh^2 r\,d\phi\,dt)\,.$$

In order to find some properties of Gödel's universe we consider its embedding in the 10-dimensional pseudoeuclidean space (2.1), given by the following formulas:

(5.3) $$\begin{aligned}&z_1=2t\,,\quad z_2=2f\cos\phi\,,\quad z_3=2f\sin\phi\,,\\ &z_4=2(2\sqrt{2})^{1/2}\sinh r\cos\tfrac{1}{2}(\phi+t)\,,\\ &z_5=2(2\sqrt{2})^{1/2}\sinh r\sin\tfrac{1}{2}(\phi+t)\,,\\ &z_6=2f\,,\quad z_7=2r\,,\quad z_8=2z\,,\\ &z_9=2(2\sqrt{2})^{1/2}\sinh r\cos\tfrac{1}{2}(\phi-t)\,,\\ &z_{10}=2(2\sqrt{2})^{1/2}\sinh r\sin\tfrac{1}{2}(\phi-t)\,,\end{aligned}$$

where $f^2=\sinh^4 r-\sinh^2 r$.

Notice that the ϕ-curves are closed curves, which will be time-like if

(5.4) $$\sinh r>1$$

and they will be null curves if $\sinh r=1$.

Hence, *the ϕ-curves which satisfy the condition* (5.4) *are closed time-like curves of length* $2\pi\sinh r\,(\sinh^2 r-1)^{1/2}$.

For a ϕ-curve (r, t, z: constants) be a geodesic it is necessary and sufficient that $(2\,\text{sihn}^3 r-\sinh r)\cosh r=0$. Hence, *from the ϕ-curves, those for which* $\sinh r=1/\sqrt{2}$ *are closed space-like geodesics.*

We seek now the geodesic curves contained in the surfaces $r=$const., $z=$const. Calling $x_1=r$, $x_2=t$, $x_3=z$, $x_4=\phi$, they satisfy the equation

(5.5) $$\{_2{}^1{}_2\}dt^2+2\{_4{}^1{}_2\}dt\,d\phi+\{_4{}^1{}_4\}d\phi^2=0,$$

which gives

(5.6) $$t=(1/\sqrt{2})(\tfrac{1}{2}-\sinh^2 r)\phi.$$

For these geodesic curves, (5.2) takes the form

(5.7) $$ds^2=(1-\tfrac{1}{2}\cosh^2 2r)d\phi^2,$$

and therefore it will be a time-like geodesic only if

(5.8) $$\cosh^2 2r<2 \quad \text{or} \quad \sinh^2 r<(\sqrt{2}-1)/2.$$

Excepting the case $\sinh r=1/\sqrt{2}$, already considered, the geodesics (5.6) are not closed. If $\sinh r<1/\sqrt{2}$, t is an increasing function of ϕ and if $\sinh r>1/\sqrt{2}$, t is a decreasing function of ϕ. Hence, in order that the point corresponding to $\phi_0+2\pi$ precedes the point corresponding to ϕ_0, it is necessary that $\sinh r>1/\sqrt{2}$ and the geodesic is space-like. The past cannot be influenced by this way.

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REFERENCES

[1] M. Abdel-Megied und G. Dautcourt: Zur Struktur des Lichtkegels im Gödel Kosmos, *Mathematische Nachrichten*, **54** (1972), 33-39.

[2] F. Bampi and Clara Zordan: A note on Gödel's metric, *General Relativity and Gravitation*, **9** (1978), 393-398.

[3] S. Chandrasekhar and J. P. Wright: The geodesics in Gödel's universe, *Proc. Nat. Acad. Sc. Washington*, **47** (1961), 341-347.

[4] L. P. Eisenhart: Riemannian geometry, *Princeton University Press*, (1949).

[5] K. Gödel: An example of a new type of cosmological solutions of Einstein's field equations of gravitation, *Reviews of Modern Physics*, **21** (1949), 447-450.

[6] K. Gödel: Rotating universes in general relativity, *Proc. Int. Congress of Math. Cambridge*, (1950), Vol. **1** (1952), 175-181.

[7] J. Rosen: Embedding of various relativistic Riemannian spaces in pseudo-euclidean spaces, *Rev. of Modern Physics*, **37** (1965), 204-214.

[8] J. L. Synge: Relativity, the general theory, *North-Holland Publ. Co., Amsterdam*, (1960).

[9] J. P. Wright: Solution of Einstein's field equations for a rotating, stationary and dust-filled universe, *J. Mathematical Physics*, **6** (1965), 103-105.