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INTEGRAL GEOMETRY

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1. INTRODUCTION

We shall begin with three simple examples which will show the basic ideas on which integral geometry has been developed.

1.1. *Sets of points.* Let X be a set of points in the euclidean plane E_2. The measure (ordinary area) of X is defined by the integral

$$m(X) = \int_X dx\, dy. \tag{1.1}$$

Let $\mathfrak{M}$ be the group of motions in E_2. With respect to an orthogonal Cartesian system of coordinates, the equations of a motion $u \in \mathfrak{M}$ are

$$\begin{aligned} x' &= x \cos \varphi - y \sin \varphi + a \\ y' &= x \sin \varphi + y \cos \varphi + b. \end{aligned} \tag{1.2}$$

The fundamental property of the measure (1.1) is that of being invariant under $\mathfrak{M}$. That is, if $X' = uX$ is the transform of X by u, we have

$$(1.3) \qquad m(X') = \int_{X'} dx'\, dy' = \int_X dx\, dy = m(X)$$

as follows immediately from (1.2). It is well known that this property characterizes the measure (1.1) up to a constant factor.

Because we are generally interested only in the differential form under the integral sign in (1.1), we shall write $dP = dx\, dy$, or, more precisely,

$$(1.4) \qquad dP = dx \wedge dy$$

to indicate that the differential form under a multiple integral sign is an exterior differential form [see, for example, Munroe (43)].

The exterior differential form (1.4) is called the density for points in E_2 with respect to $\mathfrak{M}$. We shall always take the densities in absolute value.

1.2. *Sets of lines.* Let X now be a set of lines in E_2—for example, the set of all lines G which intersect a given convex domain K. We ask for a measure of X invariant under $\mathfrak{M}$.

Let p be the distance from the origin O to G and θ the angle formed by the perpendicular to G through O and the x-axis. We maintain that this invariant measure is given by

$$(1.5) \qquad m(X) = \int_X dp\, d\theta.$$

For a proof, we observe that by the motion u [Relation (1.2)] the line coordinates p, θ transform according to

$$(1.6) \quad \theta' = \theta + \varphi, \qquad p' = p + a\cos(\theta + \varphi) + b\sin(\theta + \varphi)$$

and putting $X' = uX$, we have

$$m(X') = \int_{X'} dp'\, d\theta' = \int_X dp\, d\theta = m(X)$$

which proves the invariance of $m(X)$. That this measure is unique, up to a constant factor, follows from the transitivity of the lines under $\mathfrak{M}$, since if $\int_X f(p, \theta)\, dp\, d\theta$ is invariant we must have $\int_{X'} f(p', \theta')\, dp'\, d\theta' = \int_X f(p, \theta)\, dp\, d\theta$, and, on the other hand, according to (1.6), $\int_{X'} f(p', \theta')\, dp'\, d\theta' = \int_X f(p', \theta')\, dp\, d\theta$. From the last two equalities, we obtain $\int_X f(p', \theta')\, dp\, d\theta = \int_X f(p, \theta)\, dp\, d\theta$. If this equality holds for any set X it must be true that $f(p', \theta') = f(p, \theta)$, and, since any line $G(p, \theta)$ can be transformed into any other $G(p', \theta')$ by a motion, we deduce $f(p, \theta) =$ constant.

The differential form

$$dG = dp \wedge d\theta, \tag{1.7}$$

taken in absolute value, is called the density for lines in E_2 with respect to $\mathfrak{M}$.

Let us consider a simple application. To get the measure of the set of lines which cut a fixed segment S of length l, because of the invariance under $\mathfrak{M}$ we may take the origin of coordinates coincident with the middle point of S and the x-axis coincident with the direction of S; then we have

$$m(G; G \cap S \neq 0) = \int_{G \cap S \neq 0} dp\, d\theta = \int_0^{2\pi} \left| \frac{l}{2} \cos\theta \right| d\theta = 2l. \tag{1.8}$$

If instead of S we consider a polygonal line Γ composed of a finite number of segments S_i of lengths l_i, writing (1.8) for each S_i and summing we get

$$\int n\, dG = 2L \tag{1.9}$$

where $n = n(G)$ is the number of points in which $G(p, \theta)$ cuts Γ and L is the length of Γ. The integral in (1.9) is extended over all lines of the plane, n being 0 if $G \cap \Gamma = 0$. By a limit process it is not difficult to prove that (1.9) holds for any rectifiable curve [Blaschke (3)].

Conversely, given a continuum of points Γ in the plane, if the integral on the left of (1.9) has a meaning, then it can be taken as a definition for the length of Γ, which is the so-called Favard length [Nöbeling (45)].

For a convex curve K we have $n = 2$ for all G which intersect K, except for the positions in which G is a supporting line of K, which are of zero measure. Consequently we have: The measure of the set of lines which intersect a convex curve is equal to its length.

1.3. *Kinematic density.* Let us now consider a set X of oriented congruent segments S of length l—for example, the set of those which intersect a fixed convex domain. The position of S in E_2 is determined by the coordinates of its origin $P(x, y)$ and the angle α formed by S and the x-axis. If we want to define a measure for X invariant under $\mathfrak{M}$, we must take

$$m(X) = \int_X dx\, dy\, d\alpha. \tag{1.10}$$

To see this, we first observe that by a motion (1.2) the variables (x, y, α) transform according to (1.2) and $\alpha' = \alpha + \varphi$. Consequently the Jacobian of the transformation is 1, and we have

$$m(X') = \int_{X'} dx'\, dy'\, d\alpha' = \int_X dx\, dy\, d\alpha = m(X)$$

where $X' = uX$, which proves the invariance of $m(X)$. The uniqueness, up to a constant factor, follows from the transitivity of $\mathfrak{M}$ with respect to the congruent segments of the plane by the same argument previously given for the lines.

If instead of segments we want to measure sets of congruent figures K, since the position of such a figure is determined by the position of a point $P(x, y)$ rigidly bound to K and the angle α between a fixed direction PA in K and the x-axis, we can take the same integral (1.10). The differential form

$$dK = dx \wedge dy \wedge d\alpha \tag{1.11}$$

is called the kinematic density for E_2 with respect to the group $\mathfrak{M}$. It is always taken in absolute value.

Another form for dK is obtained if instead of the coordinates (x, y, α) for the oriented segment S, we take the coordinates (p, θ) of the line G which contains S and the distance $t = HP$ from P to the foot H of the perpendicular drawn from the origin 0 to G. The transformation formulas are

$$x = p\cos\theta + t\sin\theta, \quad y = p\sin\theta - t\cos\theta, \quad \alpha = \theta - \frac{\pi}{2} \tag{1.12}$$

and consequently, up to the sign, we have $dx \wedge dy \wedge d\alpha = dp \wedge d\theta \wedge dt$. We may then write

$$dK = d\vec{G} \wedge dt \tag{1.13}$$

where we write $\vec{G}$ in order to indicate that G must be considered as oriented ($d\vec{G} = 2\, dG$).

From this expression for dK we easily deduce the measure of the set of segments of length l which intersect a given convex domain K of area F and perimeter L. In fact, calling λ the length of the chord determined by G on K, we have

$$m(S; S \cap K \neq 0) = 2 \int dp\, d\theta\, dt = 2 \int_{S \cap K \neq 0} (\lambda + l)\, dp\, d\theta$$
$$= 2\pi F + 2lL,$$

This formula can be generalized to surfaces [see (55)]; an application was given by Green (22).

If we ask for the measure of the set of segments S which are contained in K, the result is not simple; it depends largely on K. For instance, for a circle C of diameter $D \geqq l$, we have

$$m(S; S \subset C) = \frac{\pi}{2}\left(\pi D^2 - 2D^2 \arcsin \frac{l}{D} - 2l\sqrt{D^2 - l^2}\right)$$

and for a rectangle R of sides a, b ($a \geqq l$, $b \geqq l$), we have

$$m(S; S \subset R) = 2(\pi ab - 2(a + b)l + l^2).$$

An unsolved problem is that of finding among all convex domains K with a given perimeter those which maximize the measure $m(S; S \subset K)$ of the segments of a given length which are contained in K. For $l = 0$ the problem is the classical isoperimetric problem and the solution is well-known to be the circle.

The preceding very simple examples show the three steps which constitute the so-called integral geometry in the original sense of Blaschke (3): (1) definition of a measure for sets of geometric objects with certain properties of invariance; (2) evaluation of this measure for some particular sets; and (3) application of the obtained result to get some statements of geometrical interest.

The same examples show the basic elements which are necessary to build the integral geometry from a general point of view: (1) a base space E in which the objects we consider are imbedded (in the preceding examples, E was the euclidean plane E_2); (2) a group of transformations $\mathfrak{G}$ operating on E (in the preceding examples $\mathfrak{G}$ was $\mathfrak{M}$; (3) geometric objects F contained in E which transform transitively by $\mathfrak{G}$ (in the preceding examples, the geometric objects were points, lines or congruent figures).

Given E, $\mathfrak{G}$, and F, the first problem of the integral geometry is to find a measure for sets of F invariant under $\mathfrak{G}$.

2. GENERAL INTEGRAL GEOMETRY

2.1. *Density and measure for groups of matrices.* Though the integral geometry deals with general Lie groups, from the geometrical point of view in which we are principally interested it suffices to consider Lie groups which admit a faithful representation, that is, which are isomorphic to a matrix group. We need some facts about groups of matrices, which we shall compile in this section. For a more general treatment, see Chevalley (12).

Let $\mathfrak{G}$ be a group of $n \times n$ matrices of dimension r, that is, each matrix $u \in \mathfrak{G}$ depends on r independent parameters $a_1, a_2, \cdots, a_r$; more precisely, each matrix $u \in \mathfrak{G}$ is determined by a point $a = (a_1, a_2, \cdots, a_r)$ of a differentiable manifold of dimension r, which we shall denote by the same letter $\mathfrak{G}$; $a_1, a_2, \cdots, a_r$ are then the coordinates of a in a suitable local coordinate system.

Let $e \in \mathfrak{G}$ be the unit matrix and u^{-1} the inverse of $u \in \mathfrak{G}$. If du denotes the differential of the matrix u, the equation

$$u^{-1}(u + du) = e + \omega \tag{2.1}$$

defines a matrix $\omega = u^{-1}\,du$ of linear (pfaffian) differential forms which is called the matrix of Maurer-Cartan of $\mathfrak{G}$. The elements ω_{ij} of ω have the form $\omega_{ij} = \alpha_{ij1}\,da_1 + \cdots + \alpha_{ijr}\,da_r$, where the coefficients α_{ijk} are analytic functions of $a_1, a_2, \cdots, a_r$. From these n^2 pfaffian forms ω_{ij} there are r linearly independent (base of the vector space dual of the tangent space of $\mathfrak{G}$) which we shall denote by $\omega_1, \omega_2, \cdots, \omega_r$; they are called the forms of Maurer-Cartan of $\mathfrak{G}$ and are defined up to a linear combination with constant coefficients.

The fundamental property of the matrix ω is that of being left invariant under $\mathfrak{G}$. For if $u' = su$ (s is a fixed element of $\mathfrak{G}$), we have $du' = s\,du$, and therefore $\omega' = u'^{-1}\,du' = u^{-1}\,s^{-1}\,s\,du = u^{-1}\,du = \omega$.

As a consequence, the r forms of Maurer-Cartan are also left invariant under $\mathfrak{G}$, and this fact characterizes these forms up to a linear combination with constant coefficients. For a proof, we observe that since the forms of Maurer-Cartan $\omega_1, \cdots, \omega_r$ are independent, each pfaffian form Ω may be written $\Omega(a, da) = \sum_1^r A_i(a)\omega_i$. If Ω is left invariant under $\mathfrak{G}$, we have

$$\Omega' = \textstyle\sum_1^r A_i(a')\omega_i' = \sum_1^r A_i(a)\omega_i$$

and since $\omega_i' = \omega_i$, we have

$$\textstyle\sum_1^r (A_i(a') - A_i(a))\omega_i = 0.$$

Because of the independence of ω_i, it follows that $A_i(a') = A_i(a)$, which implies A_i = constant. (Since we are interested only in the left invariance, we shall hereafter speak simply of invariance, understanding that it means left invariance.)

Notice that by exterior differentiation of $\omega = u^{-1}\,du$, taking into account that $du^{-1} = -u^{-1}\,du\,u^{-1}$, we get

$$d\omega = -u^{-1}\,du\,u^{-1} \wedge du = -\omega \wedge \omega. \tag{2.2}$$

This matric equation includes the expression of the exterior differentials $d\omega_i$ of the forms of Maurer-Cartan as linear combinations with constant coefficients of the products $\omega_j \wedge \omega_k$; these expressions are called the equations of structure of Maurer-Cartan for the group $\mathfrak{G}$.

2.2. *Density and measure in homogeneous spaces.* Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$ of dimension $r - h$. Suppose that $\mathfrak{H}$ itself is a Lie group isomorphic to a matrix group. We want to find the conditions for the existence of a density (that is, an element of volume) in the homogeneous space $\mathfrak{G}/\mathfrak{H}$ (= set of left cosets $s\mathfrak{H}$, $s \in \mathfrak{G}$) invariant under $\mathfrak{G}$. For this purpose, we notice that the submanifold $\mathfrak{H}$ of the differentiable manifold $\mathfrak{G}$ and its left cosets $s\mathfrak{H}$ ($s \in \mathfrak{G}$) are the integral manifolds of a pfaffian system.

$$\omega_1 = 0, \qquad \omega_2 = 0, \qquad \cdots, \qquad \omega_h = 0. \tag{2.3}$$

Because $\mathfrak{H}$ and its left cosets as a whole are invariant under $\mathfrak{G}$, the left side members of (2.3) will be linear combinations with constant coefficients of the forms of Maurer-Cartan of $\mathfrak{G}$, and, because these forms are defined up to a linear combination with constant coefficients, we may assume that they are the h first forms of Maurer-Cartan of $\mathfrak{G}$.

Because ω_i is invariant under $\mathfrak{G}$, the differential form

$$\Omega_h = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_h \tag{2.4}$$

will be also invariant under $\mathfrak{G}$. However, Ω_h is not always a density for $\mathfrak{G}/\mathfrak{H}$ because its value can change when the points $a \in \mathfrak{G}$ displace on the manifolds $s\mathfrak{H}$. We shall now prove the following theorem.

THEOREM: *A necessary and sufficient condition for Ω_h to be a density for $\mathfrak{G}/\mathfrak{H}$ is that its exterior differential vanish, that is,*

$$d\Omega_h = 0. \tag{2.5}$$

Proof: To prove this theorem, we observe that the submanifold $\mathfrak{H}$ and its left cosets fill up the manifold $\mathfrak{G}$ in such a way that for each point of $\mathfrak{G}$ passes one and only one submanifold. Thus, the system (2.3) is completely integrable and it is consequently equivalent to a system of the form

$$d\xi_1 = 0, \qquad d\xi_2 = 0, \qquad \cdots, \qquad d\xi_h = 0, \tag{2.6}$$

where $\xi_i = \xi_i(a_1, a_2, \cdots, a_r)$ are functions of a_i such that the manifolds $s\mathfrak{H}$ are represented by $\xi_i = \text{constant}$ ($i = 1, 2, \cdots, h$). We can make in $\mathfrak{G}$ the change of local coordinates $(a_1, a_2, \cdots, a_r) \to (\xi_1, \xi_2, \cdots, \xi_h, x_{h+1}, \cdots, x_r)$. Since the systems (2.3) and (2.6) are equivalent, we have

$$\Omega_h = A(\xi, x)\, d\xi_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_h, \tag{2.7}$$

where $A(\xi, x)$ denotes a function of $\xi_1, \cdots, \xi_h, x_{h+1}, \cdots, x_r$. When the point $a(\xi_1, \xi_2, \cdots, \xi_h, x_{h+1}, \cdots, x_r)$ varies on $s\mathfrak{H}$, the coordinates ξ_i are constant, and, therefore,

$$\delta\Omega_h = \sum_{j=h+1}^{r} \frac{\partial A}{\partial x_j} dx_j \wedge d\xi_1 \wedge \cdots \wedge d\xi_h. \tag{2.8}$$

On the other side, by exterior differentiation of (2.7), we get

$$d\Omega_h = \sum_{j=1}^{h} \frac{\partial A}{\partial \xi_j} d\xi_j \wedge d\xi_1 \wedge \cdots \wedge d\xi_h$$
$$+ \sum_{j=h+1}^{r} \frac{\partial A}{\partial x_j} dx_j \wedge d\xi_1 \wedge \cdots \wedge d\xi_h = \delta\Omega_h,$$

because the first sum vanishes. Consequently, so that $\delta\Omega_h = 0$—that is, for Ω_h to be invariant by displacements on the manifolds $s\mathfrak{H}$, it is necessary and sufficient that $d\Omega_h = 0$. This proves the theorem.

If $\mathfrak{H}$ reduces to the identity, then $\mathfrak{G}/\mathfrak{H} = \mathfrak{G}$ and $\Omega_r = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_r$ gives the invariant density (= element of volume) of $\mathfrak{G}$, which in integral geometry takes the name of kinematic density of $\mathfrak{G}$. The integral of Ω_r gives an invariant measure for $\mathfrak{G}$ (Haar's measure) which is unique up to a constant factor.

2.3. *The examples of the introduction.* To exemplify these general results, we shall consider the examples appearing in the introduction.

The group of motions $\mathfrak{G} = \mathfrak{M}$ in E_2 can be represented by the group of 3-dimensional matrices,

$$u = \begin{pmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

with the parameters $a_1 = a$, $a_2 = b$, $a_3 = \varphi$. We have

$$u^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & -b\sin\varphi - a\cos\varphi \\ -\sin\varphi & \cos\varphi & -b\cos\varphi + a\sin\varphi \\ 0 & 0 & 1 \end{pmatrix}$$

$$du = \begin{pmatrix} -\sin\varphi\, d\varphi & -\cos\varphi\, d\varphi & da \\ \cos\varphi\, d\varphi & -\sin\varphi\, d\varphi & db \\ 0 & 0 & 0 \end{pmatrix}$$

and, therefore,

$$\omega = u^{-1}\,du = \begin{pmatrix} 0 & -d\varphi & \cos\varphi\,da + \sin\varphi\,db \\ d\varphi & 0 & -\sin\varphi\,da + \cos\varphi\,db \\ 0 & 0 & 0 \end{pmatrix}$$

The forms of Maurer-Cartan are

(2.10)
$$\omega_1 = \cos\varphi\,da + \sin\varphi\,db, \quad \omega_2 = -\sin\varphi\,da + \cos\varphi\,db, \quad \omega_3 = d\varphi,$$

and the equations of structure

$$d\omega = -\omega \wedge \omega = -\begin{pmatrix} 0 & 0 & -\omega_3 \wedge \omega_2 \\ 0 & 0 & \omega_3 \wedge \omega_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is,

(2.11) $$d\omega_1 = -\omega_2 \wedge \omega_3, \qquad d\omega_2 = -\omega_3 \wedge \omega_1, \qquad d\omega_3 = 0.$$

The kinematic density of $\mathfrak{M}$ is

$$dK = \omega_1 \wedge \omega_2 \wedge \omega_3 = da \wedge db \wedge d\varphi,$$

which, up to the notation, coincides with (1.11).

Let $\mathfrak{H}_1$ be the subgroup of $\mathfrak{M}$ consisting of all motions which leave the line $G(p, \theta)$ invariant (equation of $G\colon x\cos\theta + y\sin\theta - p = 0$). There is a bijective mapping between the lines G of E_2 and the points of the space $\mathfrak{M}/\mathfrak{H}_1$. As density for lines, we take the density of $\mathfrak{M}/\mathfrak{H}_1$.

By the change of coordinates $(a, b, \varphi) \to (p, \theta, t)$ in $\mathfrak{M}$, given by the equations,

$$a = p\cos\theta + t\sin\theta, \qquad b = p\sin\theta - t\cos\theta, \qquad \varphi = \theta - \frac{\pi}{2}$$

$$p = a\cos\theta + b\sin\theta, \qquad t = a\sin\theta - b\cos\theta, \qquad \theta = \varphi + \frac{\pi}{2},$$

the points of $\mathfrak{M}/\mathfrak{H}_1$ are p = constant, θ = constant. The system (2.6) is $dp = 0$, $d\theta = 0$, and the system (2.3) is

$$dp = \cos\theta\,da + \sin\theta\,db = -\sin\varphi\,da + \cos\varphi\,db = \omega_2 = 0,$$
$$d\theta = d\varphi = \omega_3 = 0.$$

Therefore, the density for lines takes the form

(2.12) $$dG = \omega_2 \wedge \omega_3 = -\sin\varphi\,da \wedge d\varphi + \cos\varphi\,db \wedge d\varphi$$

which is equivalent to

$$dG = dp \wedge d\theta, \tag{2.13}$$

as stated in (1.7).

If $\mathfrak{H}_0$ is the subgroup of $\mathfrak{M}$ consisting of all motions which leave the point $P(a, b)$ invariant, there is a bijective mapping between the points (a, b) of E_2 and the points of the homogeneous space $\mathfrak{M}/\mathfrak{H}_0$. The system (2.6) is now $da = 0$, $db = 0$, and (2.3) gives $\omega_1 = 0$, $\omega_2 = 0$. The density (2.4) for points results in

$$dP = \omega_1 \wedge \omega_2 = da \wedge db, \tag{2.14}$$

which coincides with (1.4). In both cases (2.13) and (2.14), the condition (2.5) is obviously satisfied.

To give an example in which the homogeneous space $\mathfrak{G}/\mathfrak{H}$ has not an invariant density, let us consider the 4-dimensional group $\mathfrak{G}$ of matrices of the form

$$u = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_1 a_3 \neq 0,$$

and the 2-dimensional subgroup $\mathfrak{H}$ of matrices of the form

$$u_1 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_1 a_3 \neq 0.$$

To obtain the forms of Maurer-Cartan of $\mathfrak{G}$, we have

$$\omega = u^{-1}\,du = \begin{pmatrix} a_1^{-1} & 0 & -a_1^{-1}a_2 \\ 0 & a_3^{-1} & -a_3^{-1}a_4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} da_1 & 0 & da_2 \\ 0 & da_3 & da_4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_1 & 0 & \omega_2 \\ 0 & \omega_3 & \omega_4 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\omega_1 = a_1^{-1}\,da_1, \qquad \omega_2 = a_1^{-1}\,da_2, \qquad \omega_3 = a_3^{-1}\,da_3, \qquad \omega_4 = a_3^{-1}\,da_4.$$

The subgroup $\mathfrak{H}$ is characterized by $a_2 = 0$, $a_4 = 0$, and, therefore, the system (2.3) is now $\omega_2 = 0$, $\omega_4 = 0$. The differential form

$\Omega_2 = \omega_2 \wedge \omega_4$ is not a density, because $d\Omega_2 = -\omega_1 \wedge \omega_2 \wedge \omega_4 - \omega_3 \wedge \omega_2 \wedge \omega_4 \neq 0$.

3. INTEGRAL GEOMETRY IN THE THREE-DIMENSIONAL EUCLIDEAN SPACE

3.1. *The group of motions in E_3.* We shall consider in detail the integral geometry of the 3-dimensional euclidean space. The base space is E_3 and the group $\mathfrak{G}$ is the group of motions $\mathfrak{M}$ in it.

Let x represent the one-column matrix formed by the orthogonal coordinates x_1, x_2, x_3 of a point P. The matrix equation of a motion $x \rightarrow x'$ is

$$x' = Ax + B, \tag{3.1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{3.2}$$

and A satisfies the conditions of orthogonality

$$A^t = A^{-1} \quad (A^t = \text{transposed of } A). \tag{3.3}$$

The condition (3.3) reduces to 3 the number of independent parameters a_{ij} which, with b_1, b_2, and b_3, are the 6 parameters on which $\mathfrak{M}$ depends.

The group $\mathfrak{M}$ can be represented by the 4×4 matrices,

$$u = \left(\begin{array}{c|c} A & B \\ \hline 0 & 1 \end{array}\right) \tag{3.4}$$

with the ordinary rules,

$$u_2u_1 = \left(\begin{array}{c|c} A_2 \quad A_1 & A_2B_1 + B_2 \\ \hline 0 & 1 \end{array}\right), \quad u^{-1} = \left(\begin{array}{c|c} A^{-1} & -A^{-1}B \\ \hline 0 & 1 \end{array}\right).$$

The matrix of Maurer-Cartan is

$$\omega = u^{-1}\,du = \begin{pmatrix} A^{-1}\,dA & \vdots & A^{-1}\,dB \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & 0 \end{pmatrix}.$$

If we introduce the two matrices

$$\omega_A = A^{-1}\,dA, \qquad \omega_B = A^{-1}\,dB \tag{3.5}$$

of order 3×3 and 3×1, respectively, the equations of structure can be written

$$d\omega_A = -\omega_A \wedge \omega_A, \qquad d\omega_B = -\omega_A \wedge \omega_B. \tag{3.6}$$

Since $\mathfrak{M}$ is a 6-parameter group, we must have 6 pfaffian forms of Maurer-Cartan. Effectively, from (3.3) and (3.5) we deduce $\omega_A = A^t\,dA = -dA^t\,A = -\omega_A^t$, and the 6 forms are the elements of the matrices,

$$\omega_A = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & \omega_{33} \end{pmatrix}, \qquad \omega_B = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

which, explicitly, give

$$\omega_{ih} = -\omega_{hi} = \sum_{j=1}^{3} a_{ji}\,da_{jh}, \qquad \omega_i = \sum_{j=1}^{3} a_{ji}\,db_j. \tag{3.7}$$

It is useful to give a more geometrical approach to the pfaffian forms ω_{ih} and ω_i. Let us consider in E_3 a fixed frame $(Q_0; e_1^0, e_2^0, e_3^0)$ composed of a point Q_0 and three orthogonal unit vectors e_i^0, and a moving frame $(Q; e_1, e_2, e_3)$ which results from the fixed frame by the motion u represented by (3.1). If we introduce the matrices

$$e^0 = (e_1^0, e_2^0, e_3^0), \qquad e = (e_1, e_2, e_3) \tag{3.8}$$

we can write

$$Q = e^0\,B, \qquad e = e^0\,A, \tag{3.9}$$

and, therefore,

$$\begin{aligned} dQ &= e^0\,dB = e\,A^{-1}\,dB = e\omega_B, \\ de &= e^0\,dA = e\,A^{-1}\,dA = e\omega_A, \end{aligned} \tag{3.10}$$

which may be written

$$(3.11) \qquad dQ = \sum_{j=1}^{3} \omega_j e_j, \quad de_i = \sum_{j=1}^{3} \omega_{ji} e_j.$$

These formulas are useful for the computation of densities, as we shall see in the next section. Because of the orthogonality of the unit vectors e_i, we have $e_i e_j = \delta_{ij}$, and from (3.11) we deduce

$$(3.12) \qquad \omega_j = e_j\, dQ, \qquad \omega_{ji} = e_j\, de_i,$$

which are the vectorial form of the equations in (3.7).

3.2. *The area element of the unit sphere.* We need to remember two expressions for the element of area of the unit sphere. Let ν be the unit vector with the components

$$(3.13) \quad \nu_1 = \sin\theta\cos\varphi, \qquad \nu_2 = \sin\theta\sin\varphi, \qquad \nu_3 = \cos\theta$$

where θ, φ are the ordinary spherical coordinates corresponding to the endpoint of ν. The area element at this endpoint is known to be

$$(3.14) \qquad d\sigma = (\nu\, \nu_\theta\, \nu_\varphi)\, d\theta \wedge d\varphi = \sin\theta\, d\theta \wedge d\varphi$$

where $(\nu\, \nu_\theta\, \nu_\varphi)$ denotes the scalar triple product of the vectors ν, ν_θ, and ν_φ (subscripts denote partial derivation). Taking (3.13) into account, we have also

$$(3.15) \qquad d\sigma = \frac{d\nu_2 \wedge d\nu_3}{\nu_1} = \frac{d\nu_3 \wedge d\nu_1}{\nu_2} = \frac{d\nu_1 \wedge d\nu_2}{\nu_3},$$

and since $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$, we deduce

$$d\sigma = \nu_1\, d\nu_2 \wedge d\nu_3 + \nu_2\, d\nu_3 \wedge d\nu_1 + \nu_3\, d\nu_1 \wedge d\nu_2.$$

On the other hand, if e_1, e_2, and e_3 are the 3 orthogonal unit vectors of a moving frame, we have

$$(3.16) \qquad \begin{aligned} e_1\, de_3 \wedge e_2\, de_3 &= e_1(e_{3\theta}\, d\theta + e_{3\varphi}\, d\varphi) \wedge e_2(e_{3\theta}\, d\theta + e_{3\varphi}\, d\varphi) \\ &= (e_1 e_{3\theta} \cdot e_2 e_{3\varphi} - e_1 e_{3\varphi} \cdot e_2 e_{3\theta})\, d\theta \wedge d\varphi \\ &= (e_1 \wedge e_2) \cdot (e_{3\theta} \wedge e_{3\varphi})\, d\theta \wedge d\varphi \\ &= (e_3 e_{3\theta} e_{3\varphi})\, d\theta \wedge d\varphi = d\sigma \end{aligned}$$

where $d\sigma$ denotes the area element of the unit sphere corresponding to the endpoint of e_3. From (3.12) and (3.16), we get

(3.17) $$d\sigma = \omega_{13} \wedge \omega_{23}.$$

We have now at our disposal all elements necessary to find the densities for points, lines and planes of E_3 invariant under $\mathfrak{M}$.

3.3. *Density for points.* Let $\mathfrak{H}_0$ be the set of motions which leave the point $Q(b_1, b_2, b_3)$ invariant; clearly it is a subgroup of $\mathfrak{M}$. According to (3.11), to keep Q fixed we must have

$$\omega_1 = 0, \qquad \omega_2 = 0, \qquad \omega_3 = 0,$$

which is the system (2.3), and, according to (2.4), the density for points will be $\omega_1 \wedge \omega_2 \wedge \omega_3 = db_1 \wedge db_2 \wedge db_3$ [applying (3.7) and taking into account the determinant $|a_{ij}| = 1$, because the matrix $A = (a_{ij})$ is orthogonal]. In general, for the point $P(x, y, z)$, we shall have

(3.18) $$dP = dx \wedge dy \wedge dz.$$

The condition (2.5) is obviously satisfied.

3.4. *Density for planes.* Let $\mathfrak{H}_2$ be the set of motions which leave the plane $E(e_1, e_2)$ invariant; clearly it is a subgroup of $\mathfrak{M}$.

By the motions of $\mathfrak{H}_2$ the unit vector e_3 remains fixed and the point Q can only move on the plane e_1, e_2; therefore, according to (3.11), the pfaffian system which characterizes the planes is

$$\omega_3 = 0, \qquad \omega_{13} = 0, \qquad \omega_{23} = 0,$$

and the density for planes results:

(3.19) $$dE = \omega_3 \wedge \omega_{13} \wedge \omega_{23}.$$

If θ, φ are the spherical coordinates of the endpoint of e_3, (3.14) and (3.17) give

(3.20) $$\omega_{13} \wedge \omega_{23} = d\sigma = \sin\theta \, d\theta \wedge d\varphi.$$

If p is the distance from the origin Q_0 of the fixed frame to the plane E, and $a_{13} = \sin\theta\cos\varphi$, $a_{23} = \sin\theta\sin\varphi$, $a_{33} = \cos\theta$ are the components of e_3 (normal to E), we have $p = a_{13}b_1 + a_{23}b_2 + a_{33}b_3$, and, according to (3.7),

(3.21) $$\omega_3 = \sum_{j=1}^{3} a_{j3}\, db_j = dp + R\, d\theta + S\, d\varphi.$$

Here, R, S are functions of θ, φ, b_i, the explicit form of which has no interest for us. From (3.19) and (3.20) we get

$$dE = \sin\theta \, dp \wedge d\theta \wedge d\varphi = dp \wedge d\sigma. \tag{3.22}$$

The condition (2.5) is obviously satisfied, and hence we have: If a plane E is determined by its normal e_3 and its distance p to a fixed origin, the density is given by (3.22), where $d\sigma$ denotes the area element of the unit sphere corresponding to the endpoint of the unit vector e_3.

As an exercise, prove that if the plane is given by the equation $ux + vy + wz + 1 = 0$, its density takes the form

$$dE = \frac{du \wedge dv \wedge dw}{(u^2 + v^2 + w^2)^2}.$$

Example

Let S be a fixed segment of length l. To compute the measure of the set of planes E which intersect S, we take S on the e_3^0-axis and the middle point of S as the origin of coordinates. Then we have

$$\begin{aligned} m(E; E \cap S \neq 0) &= \int_{E \cap S \neq 0} dE \\ &= \frac{l}{2} \int_0^{2\pi} d\varphi \int_0^{\pi} |\cos\theta| \sin\theta \, d\theta = \pi l. \end{aligned} \tag{3.23}$$

If Γ is a polygonal line of length L, writing (3.23) for all sides of Γ and adding, we obtain

$$\int n \, dE = \pi L, \tag{3.24}$$

where n denotes the number of intersection points of E with Γ. By a limit process it is not difficult to prove that (3.24) holds for any rectifiable curve. The integral in (3.24) is extended over all planes of E_3, n being 0 for the planes which do not intersect Γ.

3.5. *Density for straight lines.* Let $\mathfrak{H}_1$ be the set of motions leaving the line G which contains the unit vector e_3 invariant; clearly $\mathfrak{H}_1$ is a subgroup of $\mathfrak{M}$.

By a motion of $\mathfrak{H}_1$, the point Q can only move in the direction of e_3, and, therefore, (3.11) gives $\omega_1 = 0$, $\omega_2 = 0$. Moreover, because e_3 is fixed, from (3.11) we deduce $\omega_{13} = 0$, $\omega_{23} = 0$. The pfaffian system (2.3) for the lines of E_3 becomes

$$\omega_1 = 0, \qquad \omega_2 = 0, \qquad \omega_{13} = 0, \qquad \omega_{23} = 0, \tag{3.25}$$

and the density for lines is

$$dG = \omega_1 \wedge \omega_2 \wedge \omega_{13} \wedge \omega_{23}. \tag{3.26}$$

According to (3.12), $\omega_1 \wedge \omega_2$ equals the area element of the plane (e_1, e_2) at the point Q, and we have seen that $\omega_{13} \wedge \omega_{23}$ is the area element of the unit sphere corresponding to the endpoint of e_3, that is, to the direction of G. If G is determined by its direction e_3 and its intersection point (x, y) with a fixed plane, denoting by ψ the angle between e_3 and the normal to the fixed plane, we have $\omega_1 \wedge \omega_2 = |\cos \psi|\, dx \wedge dy$, and we can write (3.26) in the form

$$dG = |\cos \psi|\, dx \wedge dy \wedge d\sigma. \tag{3.27}$$

From (3.26) and (3.6) it is easy to show that the condition (2.5) is satisfied.

As an exercise, prove that if G is given by the equations $x = az + p$, $y = bz + q$, then its density is

$$dG = \frac{da \wedge db \wedge dp \wedge dq}{(1 + a^2 + b^2)^2}.$$

Example

Let Σ be a fixed surface of class C^1 (= with a continuous tangent plane). If P denotes a point of the intersection $G \cap \Sigma$ and df denotes the area element of Σ at P, the density for lines can be written $dG = |\cos \psi|\, df \wedge d\sigma$, where ψ denotes the angle between G and the normal to Σ at P. Fixed P, the integral of $|\cos \psi|\, d\sigma$ extended over all the lines which pass through P, gives the projection of one-half the unit sphere upon a diametral plane—that is, π. The integration of df over the whole Σ gives the area F of Σ. Therefore, taking into account that each line has been counted as many times n as it has intersection points with Σ, we get

$$\int n\, dG = \pi F, \tag{3.28}$$

where the integral is extended over all lines of E_3, n being 0 for the lines which do not intersect Σ.

3.6. *Kinematic density.* The kinematic density is

$$dK = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_{12} \wedge \omega_{13} \wedge \omega_{23}. \tag{3.29}$$

To give a geometrical interpretation to $\omega_{12} = e_1\, de_2$, we observe that if we take on the plane e_1, e_2 two fixed orthogonal unit vectors e_1^*, e_2^* and call α the angle between e_1 and e_1^*, we can write $e_1 = \cos\alpha\, e_1^* + \sin\alpha\, e_2^*$, $e_2 = -\sin\alpha\, e_1^* + \cos\alpha\, e_2^*$; therefore, $e_1\, de_2 = -d\alpha$. That is, ω_{12} means an elementary rotation about the e_3-axis. Consequently, according to (3.17) and (3.29), if a motion is determined by the position of the moving frame $(Q; e_1, e_2, e_3)$, the kinematic density has the form

$$dK = dP \wedge d\sigma \wedge d\alpha, \tag{3.30}$$

where dP is the volume element of E_3 at the origin Q of the moving frame, $d\sigma$ is the area element of the unit sphere corresponding to the endpoint of e_3, and $d\alpha$ is the element of rotation about e_3. We remember that we always consider the densities in absolute value; thus, there is no question of sign.

Let us do an application of (3.30). Let Γ be a fixed curve with continuous tangent at every point and finite length L and let Σ be a moving surface of class C^1 and finite area F. Let Q be a point of $\Gamma \cap \Sigma$ and let e_3 be the normal to Σ at Q. If θ denotes the angle between e_3 and the tangent to Γ at Q (which we may take as the e_3^0-axis of the fixed frame) and df denotes the area element of Σ at Q, we have $dP = |\cos\theta|\, df \wedge ds$ (s = arc length of Γ). Putting this value in (3.30) and integrating over all the positions of Σ in which it has common point with Γ, because each position of Σ will be counted as many times n as intersection points have Σ and Γ, we get

$$\int n\, dK = 4\pi^2 FL. \tag{3.31}$$

Notice that the same formula holds if we suppose Σ fixed and Γ moving with density dK.

If Σ is the unit sphere, we can take the origin of the moving frame at the center of Σ; then we have $\int n\, dK = 8\pi^2 \int n\, dP$, and (3.31) gives

$$\int n\, dP = 2\pi L, \tag{3.32}$$

which is valid for any rectifiable curve (51).

3.7. *A differential formula.* In Section 5 we will need an important auxiliary formula which derives from (3.30). Let Σ_0 be a fixed surface of class C^1. At each point Q of Σ_0 we consider an orthogonal frame $(Q; e_1^0, e_2^0, e_3^0)$ with origin at Q and with e_3^0 normal to Σ_0. If the displacement vector on Σ_0 at Q is $\omega_1 e_1^0 + \omega_2 e_2^0$, the area element is $df = \omega_1 \wedge \omega_2$. To the unit vector e^0 tangent to Σ_0 at Q which forms with e_1^0 the angle τ_0, is attached the differential form $dL_0 = \omega_1 \wedge \omega_2 \wedge d\tau_0$ called the density for line elements $(Q; e^0)$ on Σ_0, and the pfaffian form $ds = \cos \tau_0\, \omega_1 + \sin \tau_0\, \omega_2$ called the element of length corresponding to the direction e^0.

Now let Σ_1 be a moving surface of class C^1, and assume that the intersection $\Sigma_0 \cap \Sigma_1$ is a rectifiable curve Γ. Let Q be a point of Γ and $(Q; e_1, e_2, e_3)$ be an orthogonal frame with e_3 perpendicular to Σ_1. Let ds be the length element of Γ at Q and ds_0, ds_1 those normal to Γ on Σ_0 and Σ_1, respectively. Let θ be the angle between the normals e_3^0, e_3. If df_0, df_1 are the elements of area of Σ_0, Σ_1 at Q and dP denotes the element of volume of E_3 at Q, we have $dP = \sin \theta\, df_0 \wedge ds_1$ and $df_1 = ds \wedge ds_1$. The element of area of the unit sphere at the endpoint of e_3 may be written $d\sigma = \sin \theta\, d\theta \wedge d\tau_0$. Putting now $d\tau_1 = d\alpha$ to unify the notation of (3.30), from this equation and the preceding relation, we deduce immediately (up to the sign)

$$\begin{aligned} ds \wedge dK &= \sin^2 \theta\, df_0 \wedge d\tau_0 \wedge df_1 \wedge d\tau_1 \wedge d\theta \\ &= \sin^2 \theta\, dL_0 \wedge dL_1 \wedge d\theta, \end{aligned} \tag{3.33}$$

which is the differential formula we want.

An immediate consequence is obtained by integrating both sides over all positions of the moving surface Σ_1. We get

$$\int L\, dK = 2\pi^3 F_0 F_1, \tag{3.34}$$

where L denotes the length of the curve $\Sigma_0 \cap \Sigma_1$, and F_0, F_1 are the areas of Σ_0, Σ_1, respectively.

If Σ_1 is the unit sphere and we take the origin of the moving frame at the center of Σ_1, we have

$$\int L\, dK = 8\pi^2 \int L\, dP_1$$

and (3.34) gives

$$\int L\, dP = \pi^2 F_0. \tag{3.35}$$

3.8. *A definition of area.* Let now Σ_1, Σ_2 be two moving unit spheres and Σ_0 a fixed surface. Let N be the number of points of the intersection $\Sigma_0 \cap \Sigma_1 \cap \Sigma_2$. If dP_i denotes the volume element at the center of Σ_i ($i = 1, 2$), we get from (3.32) and (3.35)

$$\int N\, dP_1\, dP_2 = 2\pi \int L\, dP_1 = 2\pi^3 F_0.$$

Conversely, this result conduces to define the area of a continuum of points by the formula

$$F_0 = \frac{1}{2\pi^3} \int N\, dP_1\, dP_2,$$

provided the integral of the right-hand side exists [see (52)]. Applications of the integral geometry to the definition of area for k-dimensional surfaces have been made by Federer (17–19) and Hadwiger (23) and (25). See also Nöbeling (45) and (46).

3.9. *Planes through a fixed point.* Let us now consider the set of planes E_0 which pass through a fixed point O. The density for sets of E_0 invariant under the group $\mathfrak{M}_0$ of the rotations about 0, is clearly $dE_0 = d\sigma$, where $d\sigma$ denotes the area element of the unit sphere corresponding to the direction perpendicular to E_0. In fact, this differential form is invariant under $\mathfrak{M}_0$, and, because of the transitivity of the planes E_0 with respect to $\mathfrak{M}_0$, it is unique up to a constant factor. The planes E_0 are considered non-oriented; therefore, the measure of all the planes through O will be

$$\int dE_0 = \int_{\frac{1}{2}Z} d\sigma = 2\pi, \tag{3.36}$$

where $\frac{1}{2}Z$ denotes the half of the unit sphere.

Let S be a fixed arc of great circle on the unit sphere of center O of length α. The measure of the set of planes E_0 which intersect S (= measure of the set of great circles which intersect S) will be the area of the lune bounded by the great circles the poles of which are the endpoints of S—that is, $m(E_0; S \cap E_0 \neq 0) = 2\alpha$. If instead of S we have a spherical polygonal line Γ the sides of which have the lengths α_i, we have, writing the last formula for each side and adding,

$$\int n \, dE_0 = 2L, \tag{3.37}$$

where L denotes the total length of Γ. The integration is extended over all (non-oriented) planes through O—that is, according to $dE_0 = d\sigma$, over half the unit sphere. By a limit process we can prove that (3.37) holds for any rectifiable spherical curve of the unit sphere.

Following Fenchel (20), we want to apply (3.37). Let K be a closed space curve of class C^2 without multiple points and let Γ be the spherical indicatrix of it (= the curve $T = T(s)$, where T is the tangent unit vector to K). The arc length element of Γ is $ds_t = |\varkappa| \, ds$, where $\varkappa$ denotes the curvature and s the length of K. Consequently, (3.37) yields

$$\int n \, dE_0 = 2 \int_K |\varkappa| \, ds. \tag{3.38}$$

Every closed space curve K has at least 2 tangents which are parallel to an arbitrary plane. This means that every plane E_0 intersects Γ in at least 2 points. Hence, $n \geqq 2$, and (3.36) and (3.38) give

$$\int_K |\varkappa| \, ds \geqq 2\pi, \tag{3.39}$$

a classical inequality of Fenchel.

If K is knotted, it is easy to see that it has at least 4 tangents parallel to an arbitrary plane. Hence, $n \geqq 4$, and (3.36) and (3.38) give the following inequality of Fáry (for knotted curves) (16),

$$\int_K |\varkappa| \, ds \geqq 4\pi. \tag{3.40}$$

These results have been generalized to closed varieties in E_n by Chern and Lashof (11).

4. APPLICATIONS TO CONVEX BODIES

The integral geometry is closely related to the theory of convex bodies. We compile in this section some simple facts on this theory from many sources—for example, Bonnesen and Fenchel (4), Busemann (5), Hadwiger (24), and Vincensini (72).

Let k be a plane convex set of area f placed in E_3. Let f_σ be the area of the orthogonal projection of k on a plane perpendicular to the direction σ, and let θ be the angle between σ and the normal to the plane which contains k; we have $f_\sigma = |\cos\theta|\, f$. If $d\sigma$ denotes the area element of the unit sphere Z corresponding to the direction σ, we have

$$\int_Z f_\sigma\, d\sigma = f\int_0^{2\pi} d\varphi \int_0^\pi |\cos\theta| \sin\theta\, d\theta = 2\pi f, \tag{4.1}$$

and, therefore,

$$f = \frac{1}{2\pi}\int_Z f_\sigma\, d\sigma. \tag{4.2}$$

Now let K be a convex body of E_3; we shall denote by ∂K the convex surface bounding K. Let F be the area of ∂K and F_σ the area of the orthogonal projection of K on a plane perpendicular to the direction σ. Applying (4.2) to each element of area of ∂K and integrating over all ∂K, we get

$$F = \frac{1}{\pi}\int_Z F_\sigma\, d\sigma, \tag{4.3}$$

known as Cauchy's formula for the area of a convex body.

Let O be an interior point of K and $p = p(\sigma) = p(\theta, \varphi)$ be the supporting function of K with respect to O (= distance from O to the supporting plane perpendicular to the direction σ of spherical coordinates θ, φ). The convex body K_h parallel to K at distance h has the supporting function $p_h = p(\sigma) + h$, and if R_1, R_2 are the principal radii of curvature of ∂K, those of ∂K_h at corresponding points are $R_1 + h$ and $R_2 + h$. Between the area element df of

∂K and the area element $d\sigma$ of its spherical image, there is the relation $df/d\sigma = R_1R_2$, and consequently, we have

$$(4.4) \qquad F = \int_Z R_1R_2\, d\sigma.$$

Applying this formula to ∂K_h, we get

$$(4.5) \qquad F_h = \int_Z (R_1 + h)(R_2 + h)\, d\sigma = F + 2Mh + 4\pi h^2,$$

where

$$(4.6) \qquad M = \frac{1}{2}\int_Z (R_1 + R_2)\, d\sigma = \frac{1}{2}\int_{\partial K} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) df$$

is the integral of mean curvature of ∂K. If V denotes the volume of K and V_h that of K_h, from (4.5) we deduce

$$(4.7) \qquad V_h = V + \int_0^h F_h\, dh = V + Fh + Mh^2 + \tfrac{4}{3}\pi h^3,$$

which is the so-called Steiner's formula for parallel convex bodies in E_3.

For plane convex sets, the formula analogous to (4.7) is

$$(4.8) \qquad f_h = f + uh + \pi h^2,$$

where u = length of ∂k. Applying (4.8) to the orthogonal projection of K on a plane perpendicular to the direction σ, we have

$$F_{\sigma,h} = F_\sigma + u_\sigma h + \pi h^2,$$

and by Cauchy's formula,

$$(4.9) \qquad F_h = \frac{1}{\pi}\int_Z F_{\sigma,h}\, d\sigma = \frac{1}{\pi}\int_Z F_\sigma\, d\sigma + \frac{h}{\pi}\int_Z u_\sigma\, d\sigma + 4\pi h^2.$$

Comparing (4.9) with (4.5), we get (since both formulas hold for any h)

$$(4.10) \qquad M = \frac{1}{2\pi}\int_Z u_\sigma\, d\sigma,$$

which is a very useful expression for the integral of mean curvature of the boundary of a convex body.

On the other side, considering the volume V of K as a sum of pyramids with the common vertex O, we have

(4.11) $$V = \tfrac{1}{3}\int_{\partial K} p\, df = \tfrac{1}{3}\int_Z pR_1R_2\, d\sigma.$$

Applying this formula to K_h, we get

$$V_h = \frac{1}{3}\int_Z (p+h)(R_1+h)(R_2+h)\, d\sigma$$

$$= V + \frac{h}{3}\int_Z (R_1R_2 + p(R_1+R_2))\, d\sigma$$

$$+ \frac{h^2}{3}\int_Z (p + R_1 + R_2)\, d\sigma + \frac{4}{3}\pi h^3.$$

Comparison with (4.7) yields

(4.12) $$F = \tfrac{1}{2}\int_Z p(R_1+R_2)\, d\sigma, \qquad M = \int_Z p\, d\sigma.$$

The last formula allows definition of M for any convex body without the conditions of regularity necessary to define the principal radii of curvature of ∂K. A practical way to compute M for convex surfaces ∂K not sufficiently smooth is to compute the integral of mean curvature M_h of the parallel surface ∂K_h (which is smooth) and then to pass to the limit for $h \to 0$. This method yields the following results easily. (1) For a convex polyhedron the edges of which have lengths a_i and the corresponding dihedral angles of which are α_i, we have

$$M = \tfrac{1}{2}\sum (\pi - \alpha_i)a_i.$$

(2) For a right cylinder of height h and radius r,

$$M = \pi h + \pi^2 r.$$

(3) For a plane convex domain, considered as a flattened convex body of E_3, we have

$$M = \frac{\pi}{2}u,$$

where u is the length of the boundary of the domain.

Notice that, according to (3.22), the second formula (4.12) gives the measure of the set of planes E which cut K—that is, we have the formula

$$\int_{E \cap K \neq 0} dE = M. \tag{4.13}$$

On the other side, applying (3.28) to convex surfaces ($n = 2$), we get

$$\int_{G \cap K \neq 0} dG = \frac{\pi}{2} F. \tag{4.14}$$

We may therefore state:

The volume V of a convex body K is the measure of the points contained in it; the area F of ∂K is (up to the constant factor $\pi/2$) the measure of the lines which intersect K; the integral of mean curvature M is the measure of the planes which intersect K.

These integral geometric interpretations of V, F, and M have been generalized to convex bodies of the n-dimensional euclidean space [(60) and Hadwiger (23) and (25)].

5. THE KINEMATIC FUNDAMENTAL FORMULA IN E_3

5.1. *The Euler characteristic of a domain.* Let Σ be a closed surface in E_3 which is of class C^2 and bounds a domain D of volume V. If df is the area element of Σ and $d\sigma$ the area element of the corresponding spherical image, we know the formulas

$$\frac{d\sigma}{df} = \frac{1}{R_1R_2}, \qquad I(\Sigma) = \int_\Sigma \frac{1}{R_1R_2}\, df = 4\pi\chi, \tag{5.1}$$

where R_1, R_2 are the principal radii of curvature, $I(\Sigma)$ denotes the area of the spherical image of Σ, and $\chi = \chi(D)$ is the Euler characteristic of D. Because Σ is closed, its spherical image covers the unit sphere an integer number of times, and therefore $\chi = I(\Sigma)/4\pi$ is an integer. For example, for domains topologically equivalent to the solid sphere, $\chi = 1$, and for domains which are topologically equivalent to a torus, $\chi = 0$ [see, for example, Struik (69, p. 159)].

If Σ is not of class C^2 but consists of a finite number of faces (= pieces of class C^2) which intersect along edges (= closed curves of class C^2), the Euler characteristic is obtained adding to the area of the spherical image of the faces (5.1) the area of the spherical image corresponding to the edges, which we shall now compute. Let Γ be an edge of Σ and let T, N, B, denote its unit vectors tangent, principal normal, and binormal; let s be the arc length of Γ. If e_3, e_3' are the outward normal unit vectors to the faces of Σ at the points of Γ and we call θ_1, θ_1' the angles which they form with $-N$, the spherical image corresponding to Γ is the portion of unit sphere defined by the equation,

$$Y(s, \theta) = -\cos\theta\, N + \sin\theta\, B \quad (\theta_1 \leqq \theta \leqq \theta_1', 0 \leqq s \leqq L),$$

where L is the length of Γ.

Using Frenet's formulas, we have $Y_\theta^2 = 1$, $Y_s Y_\theta = -\tau$, $Y_s^2 = \kappa^2 \cos^2\theta + \tau^2$, $(Y_\theta^2 Y_s^2 - (Y_s Y_\theta)^2)^{1/2} = \kappa \cos\theta$, where κ and τ are the curvature and the torsion of Γ. The area $I(\Gamma)$ of the spherical image corresponding to Γ will be

$$(5.2) \qquad I(\Gamma) = \int_\Gamma \kappa \cos\theta\, d\theta\, ds = \int_\Gamma (\sin\theta_1' - \sin\theta_1)\, \kappa ds.$$

Under the assumption that Σ has no vertices (= points in which more than two different faces intersect), the Euler characteristic of Σ is given by the second formula (5.1); we take into account that at the left side, the integral analogous to (5.2) for all the edges of Σ should be added.

5.2. *The kinematic formula.* Let D_0, D_1 be two domains of E_3 bounded respectively by the surfaces Σ_0, Σ_1, which we assume to be of class C^2. Let V_i, χ_i be the volume and the Euler characteristic of D_i and let F_i, M_i be the area and the integral of mean curvature of Σ_i ($i = 0, 1$). Suppose D_0 is fixed and D_1 is moving, and let dK be the kinematic density for D_1. If $\Phi(D_0 \cap D_1)$ denotes a function of the intersection $D_0 \cap D_1$, one of the main purposes of the integral geometry is the evaluation of integrals of the type

$$(5.3) \qquad J = \int \Phi(D_0 \cap D_1)\, dK$$

over all positions of D_1. For example, if $\Phi = V_{01}$ = volume of $D_0 \cap D_1$, we can easily prove that $\int V_{01}\, dK = 8\pi^2 V_0 V_1$, and if

$\Phi = F_{01}$ is the area of the boundary of $D_0 \cap D_1$, the formula $\int F_{01}\, dk = 8\pi^2(V_0F_1 + V_1F_0)$ holds (50). The most important case corresponds to $\Phi = \chi(D_0 \cap D_1)$ is the Euler characteristic of $D_0 \cap D_1$. Surprisingly enough, the integral $\int \chi(D_0 \cap D_1)\, dK$ over all positions of D_1 can be expressed by only V_i, χ_i, F_i, M_i $(i = 0, 1)$. The result is the following:

$$(5.4) \qquad \int \chi(D_0 \cap D_1)\, dK = 8\pi^2(V_0 x_1 + V_1\chi_0) + 2\pi(F_0M_1 + F_1M_0).$$

This result is the so-called kinematic fundamental formula, which we shall now prove.

We need to compute $\chi(D_0 \cap D_1)$. The boundary of $D_0 \cap D_1$ consists in a part Σ_{01} of Σ_0 which is interior to D_1 and a part Σ_{10} of Σ_1 which is interior to D_0. Both Σ_{01} and Σ_{10} are of class C^2 and are joined by an edge $\Gamma = \Sigma_0 \cap \Sigma_1$, composed of one or more closed curves, of the boundary of $D_0 \cap D_1$. According to (5.1) we will have

$$(5.5) \qquad 4\pi\chi(D_0 \cap D_1) = I(\Sigma_{01}) + I(\Sigma_{10}) + I(\Gamma),$$

and we can write

$$(5.6) \qquad 4\pi \int \chi(D_0 \cap D_1)\, dK = \int I(\Sigma_{01})\, dK + \int I(\Sigma_{10})\, dK + \int I(\Gamma)\, dK,$$

where the integrals are extended over all positions of D_1.

The first two integrals on the right-hand side of (5.6) are easily evaluated. Taking the first integral, let P be a point of $\Sigma_0 \cap D_1$ and let $d\sigma_P$ denote the area element of the unit sphere at the spherical image of P. By first fixing D_1 and then letting P vary over $\Sigma_0 \cap D_1$, we get

$$\int_{P \in \Sigma_0 \cap D_1} d\sigma_P\, dK = \int I(\Sigma_{01})\, dK,$$

and by first fixing P and then rotating D_1 about this point and letting it vary over D_1 and Σ_0,

$$\int_{P \in \Sigma_0 \cap D_1} d\sigma_P \, dK = \int_{P \in \Sigma_0} d\sigma_P \int_{P \in D_1} dK = 8\pi^2 V_1 \int_{P \in \Sigma_0} d\sigma_P$$
$$= 8\pi^2 V_1 I(\Sigma_0) = 32\pi^3 V_1 \chi_0.$$

Thus, we have

$$\int I(\Sigma_{01}) \, dK = 32\pi^3 V_1 \chi_0. \tag{5.7}$$

Similarly, by the evident invariance of the kinematic density under the inversion of the motion, we have

$$\int I(\Sigma_{10}) \, dK = 32\pi^3 V_0 \chi_1. \tag{5.8}$$

It remains to evaluate the third integral in (5.6). Let Q be a point of Γ. By Meusnier's theorem, if ρ is the radius of curvature of Γ and R, r are the radii of normal curvature of Σ_0 and Σ_1 in the direction of the tangent to Γ at Q, we have

$$\rho = R \cos \theta_1 = r \cos \theta_1' \tag{5.9}$$

where θ_1, θ_1' are the angles between the outward normals e_3, e_3' to Σ_0, Σ_1 at Q and the vector $-N$ opposite to the principal normal N of Γ at Q. Taking into account the identity

$$\frac{\sin \theta_1' - \sin \theta_1}{\cos \theta_1' + \cos \theta_1} = \tan \frac{1}{2} (\theta_1' - \theta_1), \tag{5.10}$$

and putting $\theta_1' - \theta_1 = \theta$, we deduce from (5.9) and (5.10)

$$\sin \theta_1' - \sin \theta_1 = \rho \left(\frac{1}{R} + \frac{1}{r} \right) \tan \frac{1}{2} \theta. \tag{5.11}$$

If τ_0, τ_1 denote the angles between the tangent to Γ at Q and the first principal direction of Σ_0, Σ_1 at Q, by Euler's theorem we have

$$\frac{1}{R} = \frac{\cos^2 \tau_0}{R_1} + \frac{\sin^2 \tau_0}{R_2}, \qquad \frac{1}{r} = \frac{\cos^2 \tau_1}{r_1} + \frac{\sin^2 \tau_1}{r_2}, \tag{5.12}$$

where R_1, R_2 are the principal radii of curvature of Σ_0, and r_1, r_2 are those of Σ_1 at Q. By (5.2), (5.11), and (3.33) we have

$$\int I(\Gamma) \, dK = \int \left(\frac{\cos^2 \tau_0}{R_1} + \frac{\sin^2 \tau_0}{R_2} + \frac{\cos^2 \tau_1}{r_1} + \frac{\sin^2 \tau_1}{r_2} \right) \tag{5.13}$$
$$\tan \tfrac{1}{2}\theta \sin^2 \theta \, df_0 \, d\tau_0 \, df_1 \, d\tau_1 \, d\theta,$$

where the limits of integration for the angles are

$$0 \leqq \tau_0 \leqq 2\pi, \qquad 0 \leqq \tau_1 \leqq 2\pi, \qquad 0 \leqq \theta \leqq \pi.$$

Computing the integral in (5.13), we get

$$\int I(\Gamma)\, dK = 8\pi^2(F_0M_1 + F_1M_0). \tag{5.14}$$

Adding (5.7), (5.8), and (5.14), and considering (5.6), we get the desired result (5.4).

The formula (5.4) is the work of Blaschke (3). It has been generalized to E_n by Chern (8). For the generalization to spaces of constant curvature (noneuclidean geometry) see Wu (76) and (54), (57), and (58). For another kind of proof valid for more general domains than those considered here, see Hadwiger (23).

Notice that if D_0, D_1 are convex bodies, we have $\chi(D_0) = \chi(D_1) = \chi(D_0 \cap D_1) = 1$ if $D_0 \cap D_1 \neq 0$, and $\chi(D_0 \cap D_1) = 0$, if $D_0 \cap D_1 = 0$. The formula (5.4) yields

$$\int_{D_0 \cap D_1 \neq 0} dK = 8\pi^2(V_0 + V_1) + 2\pi(F_0M_1 + F_1M_0), \tag{5.15}$$

which gives the measure of the set of congruent convex bodies D_1 having a common point with a fixed convex body D_0.

If D_1 is a sphere of radius r, we can take the origin of the moving frame at the center of D_1; then we have $\int dK = 8\pi^2 \int dP$, and (5.15) gives

$$\int dP = V_0 + F_0r + M_0r^2 + \tfrac{4}{3}\pi r^3,$$

the Steiner's formula (4.7).

6. INTEGRAL GEOMETRY IN COMPLEX SPACES

6.1. *The unitary group.* The integral geometry of complex spaces has not been developed very much, and it deserves further study. We shall give a simple typical example.

Let P_n be the n-dimensional complex projective space with the homogeneous coordinates $z_i (i = 0, 1, \cdots, n)$, so that $z = (z_0, z_1, z_2, \cdots, z_n)$ and $\lambda z = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$, where λ is a nonzero complex number; define the same point. Let $\bar{z}_i$ denote the complex conjugate of z_i. We assume the homogeneous coordinates z_i are normalized so that

$$(z\bar{z}) = \sum_0^n z_i\bar{z}_i = 1, \tag{6.1}$$

which determine z_i up to a factor of the form $\exp(i\alpha)$.

We consider the group $\mathfrak{U}$ (unitary group) of linear transformations

$$z' = Az \tag{6.2}$$

which leaves the form (6.1) invariant. The matrices A satisfy

$$A\bar{A}^t = E, \qquad A^{-1} = \bar{A}^t, \qquad \bar{A}^tA = E, \tag{6.3}$$

where E is the unit matrix. These relations show that $\mathfrak{U}$ depends upon $(n + 1)^2$ real parameters. If we interpret the elements a_{hk} $(h = 0, 1, \cdots, n)$ of the matrix A as the homogeneous coordinates of a point $a_k \in P_n$, the conditions (6.3) give

$$(a_j\bar{a}_k) = \delta_{jk}, \tag{6.4}$$

which show that the points a_k are normalized; they form the vertices of an autoconjugate n-simplex with respect to the quadric $(z\bar{z}) = 0$. Because a_k and $a_k \exp(i\alpha_k)$ are the same geometric point, to determine an element $u \in \mathfrak{U}$ we must give the $n + 1$ geometric points a_k [with the conditions of (6.4)], as well as the $n + 1$ real parameters α_k.

The invariant matrix of Maurer-Cartan is

$$\omega = A^{-1}\, dA = \bar{A}^t\, dA, \tag{6.5}$$

which satisfies, in consequence of (6.3),

$$\omega + \bar{\omega}^t = 0. \tag{6.6}$$

The invariant pfaffian forms are

$$\omega_{jk} = \sum_{h=0}^n \bar{a}_{hj}\, da_{hk} = (\bar{a}_j\, da_k), \tag{6.7}$$

and (6.6) gives

$$\omega_{jk} + \bar{\omega}_{kj} = 0. \tag{6.8}$$

The kinematic density of $\mathfrak{U}$, up to a constant factor, is

(6.9) $du = [\Pi\, \omega_{jk}\bar{\omega}_{jk} \,\Pi\, \omega_{hh}], \quad j < k, \quad 0 \leqq j, k, h \leqq n,$

where the product is exterior.

We have all necessary elements for the study of the integral geometry of the unitary group. We shall restrict ourselves to the case $n = 2$ (complex projective plane).

6.2. *Meromorphic curves.* A complex analytic mapping $E_1 \to P_2$ of the complex euclidean line E_1 into the complex projective plane P_2 defines a meromorphic curve in the sense of J. Weyl, H. Weyl (75), and L. Ahlfors (1); it is defined by three analytic functions $z_i = z_i(t)$, $(i = 0, 1, 2)$. Every such curve Γ has an invariant integral with respect to $\mathfrak{U}$, which we shall call the order of Γ. When the homogeneous coordinates z_i are normalized such that the condition (6.1) is satisfied, the order of Γ is defined by the following integral (up to the sign which depends upon the orientation assumed for Γ),

$$J = \frac{1}{2\pi i} \int_\Gamma \Omega \tag{6.10}$$

where $i = \sqrt{-1}$ and

$$\Omega = [dz\, d\bar{z}] = dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2. \tag{6.11}$$

If Γ is an algebraic curve, we shall see that J coincides with its ordinary order or grad.

If the coordinates z_i are not normalized, we set $Z_i = z_i/(z\bar{z})^{1/2}$, and an easy calculation gives

$$\Omega = [dZ\, d\bar{Z}] = \frac{|z \wedge z'|^2}{z^4} \quad dt \wedge d\bar{t}, \tag{6.12}$$

where $z \wedge z'$ denotes the vector with the components $z_1z_2' - z_2z_1'$, $z_2z_0' - z_0z_2'$, and $z_0z_1' - z_1z_0'$.

For some purposes, it is convenient to write Ω in another form. Let c be a point on the tangent to Γ at the point z such that

$$(c\bar{c}) = 1, \qquad (\bar{c}z) = 0. \tag{6.13}$$

We will have (since c is on the tangent to Γ at z),

$$dz = \alpha z + \beta c, \qquad d\bar{z} = \bar{\alpha}\bar{z} + \bar{\beta}\bar{c} \tag{6.14}$$

where α, β are the pfaffian forms

$$\alpha = (\bar{z}\, dz), \qquad \beta = (\bar{c}\, dz). \tag{6.15}$$

From (6.13) and (6.14), we deduce

$$\Omega = [dz\, d\bar{z}] = \alpha \wedge \bar{\alpha} + \beta \wedge \bar{\beta}, \tag{6.16}$$

and because $\alpha = -\bar{\alpha}$, we have $\alpha \wedge \bar{\alpha} = 0$. Therefore,

$$\Omega = \beta \wedge \bar{\beta} = (\bar{c}\, dz) \wedge (c\, d\bar{z}), \tag{6.17}$$

a formula which will be useful in the following discussion.

As an application, we shall use (6.17) to obtain the order of a complex straight line. Since J is invariant by unitary transformations and any line can be transformed into the axis $z_1 = 0$, it suffices to compute the order in this case. We take, in order to satisfy (6.1) and (6.13),

$$z = (\rho e^{i\varphi}(1 + \rho^2)^{-1/2}, 0, (1 + \rho^2)^{-1/2}),$$
$$c = (-e^{i\varphi}(1 + \rho^2)^{-1/2}, 0, \rho(1 + \rho^2)^{-1/2}),$$

and we get

$$(\bar{c}\, dz) = -\frac{d\rho + i\rho\, d\varphi}{1 + \rho^2}, \qquad (c\, d\bar{z}) = -\frac{d\rho - i\rho\, d\varphi}{1 + \rho^2}$$

and

$$\Omega = (\bar{c}\, dz) \wedge (c\, d\bar{z}) = \frac{2i\rho}{(1 + \rho^2)^2}\, d\varphi \wedge d\rho. \tag{6.18}$$

The order of the segment $a \leqq \rho \leqq b$, $0 \leqq \varphi \leqq 2\pi$ will be

$$J = \frac{1}{2\pi i} \int_a^b \int_0^{2\pi} \frac{2i\rho}{(1 + \rho^2)^2}\, d\varphi\, d\rho = \frac{b^2 - a^2}{(1 + a^2)(1 + b^2)}.$$

For $a = 0$, $b = \infty$, we obtain $J = 1$, which is the order of a line.

6.3. *A generalization of the theorem of Bezout.* Let Γ_1, Γ_2 be two meromorphic curves of P_2 of orders J_1, J_2, respectively. Let $u\Gamma_2$ be the transform of Γ_2 by $u \in \mathfrak{U}$. In the theory of meromorphic curves it is important to determine the difference between the product $J_1 J_2$ and the number $N(\Gamma_1 \cap u\Gamma_2)$ of points of intersection of Γ_1 and $u\Gamma_2$, each counted with its proper multiplicity [Ahlfors (1), Chern (9) and (10), and H. Weyl (75)].

Our goal is more simple. We wish to obtain the mean value of $N(\Gamma_1 \cap u\Gamma_2)$ for all $u \in \mathfrak{U}$. First, we will compute the integral

$$I = \int_{\mathfrak{U}} N(\Gamma_1 \cap u\Gamma_2)\, du \tag{6.19}$$

where the element of volume du is given by (6.9). In our case, $n = 2$, making use of (6.8), and considering only the absolute value, we have

$$du = (\bar{a}_0\, da_1) \wedge (\bar{a}_1\, da_0) \wedge (\bar{a}_0\, da_2) \wedge (\bar{a}_2\, da_0) \wedge (\bar{a}_1\, da_2) \wedge (\bar{a}_2\, da_1) \wedge (\bar{a}_0\, da_0) \wedge (\bar{a}_1\, da_1) \wedge (\bar{a}_2\, da_2). \tag{6.20}$$

Inasmuch as we are only interested in the transformations u such that $\Gamma_1 \cap u\Gamma_2 \neq 0$, we may choose the points a_0, a_1, and a_2, which determine u, so that: a_0 = point of $\Gamma_1 \cap u\Gamma_2$; a_1 = point on the tangent to $u\Gamma_2$ at a_0; a_2 is then determined by the relations (6.4), which we now write

$$(a_0\bar{a}_0) = (a_1\bar{a}_1) = (a_2\bar{a}_2) = 1,\ (a_0\bar{a}_1) = (a_0\bar{a}_2) = (a_1\bar{a}_2) = 0. \tag{6.21}$$

Let s be the point in which the line determined by a_1, a_2 intersects the tangent to Γ_1 at a_0. We shall have

$$(s\bar{s}) = 1, \qquad (s\bar{a}_0) = 0, \qquad (\bar{s}a_0) = 0. \tag{6.22}$$

According to (6.17), the differential form which gives the order of $u\Gamma_2$ is

$$\Omega_2 = (\bar{a}_1\, da_0) \wedge (a_1\, d\bar{a}_0) = (\bar{a}_0\, da_1) \wedge (\bar{a}_1\, da_0). \tag{6.23}$$

Since we always take a_0 on Γ_1, we have $da_0 = \alpha a_0 + \beta s$, where $\alpha = (\bar{a}_0\, da_0)$, $\beta = (\bar{s}\, da_0)$. Consequently, we have

$$(\bar{a}_2\, da_0) = \beta(\bar{a}_2 s), \qquad (a_2\, d\bar{a}_0) = \bar{\beta}(a_2\bar{s}),$$

and, by exterior multiplication,

$$(\bar{a}_2\, da_0) \wedge (a_2\, d\bar{a}_0) = (\bar{a}_0\, da_2) \wedge (\bar{a}_2\, da_0) = (\beta \wedge \bar{\beta})(\bar{a}_2 s)(a_2\bar{s}) = (\bar{a}_2 s)(a_2\bar{s})\Omega_1, \tag{6.24}$$

where Ω_1 is the differential form which gives the order of Γ_1.

From (6.20), (6.23), and (6.24), we have

$$du = \Omega_2 \wedge \Omega_1(\bar{a}_2 s)(a_2\bar{s}) \wedge (\bar{a}_1\, da_2) \wedge (\bar{a}_2\, da_1) \wedge (\bar{a}_0\, da_0) \wedge (\bar{a}_1\, da_1) \wedge (\bar{a}_2\, da_2). \tag{6.25}$$

We first keep fixed the geometric points a_0, a_1, and a_2. With the normalization (6.21), their homogeneous coordinates a_{hj} ($h = 0, 1, 2$) are determined up to an exponential factor $\exp(i\alpha_j)$; the parameters $\alpha_j (j = 0, 1, 2)$ are variables in (6.25). Putting $a_j = a_j^* \exp(i\alpha_j)$, we have $da_j = a_j i \, d\alpha_j$, $(\bar{a}_j \, da_j) = i \, d\alpha_j$, and, consequently, $\int (\bar{a}_j \, da_j) = 2\pi i \ (j = 0, 1, 2)$.

From the right side of (6.25) it remains to evaluate (a_0 being fixed) $\int (\bar{a}_2 s)(a_2 \bar{s})(\bar{a}_1 \, da_2) \wedge (\bar{a}_2 \, da_1)$, where a_1, a_2 describe the line $(\bar{a}_0 z) = 0$ which contains the point s. We can assume, because of the invariance of the integrand by unitary transformations, that this line is the axis $z_1 = 0$. According to (6.18), we then have

$$(6.26) \qquad \int (\bar{a}_1 \, da_2) \wedge (\bar{a}_2 \, da_1) = \int \frac{2i\rho}{(1 + \rho^2)^2} \, d\rho \, d\varphi,$$

where we have put $a_2 = (\rho e^{i\varphi}(1 + \rho^2)^{-1/2}, 0, (1 + \rho^2)^{-1/2})$, $a_1 = (-e^{i\varphi}(1 + \rho^2)^{-1/2}, 0, \rho(1 + \rho^2)^{-1/2})$. Taking $s = (0, 0, 1)$, we obtain

$$(6.27) \qquad (\bar{a}_2 s)(a_2 \bar{s}) = \frac{1}{1 + \rho^2},$$

and, therefore,

$$(6.28) \qquad \int (\bar{a}_2 s)(a_2 \bar{s})(\bar{a}_1 \, da_2) \wedge (\bar{a}_2 \, da_1) = \int_0^\infty \int_0^{2\pi} \frac{2i\rho}{(1 + \rho^2)^3} \, d\rho \, d\varphi = \pi i.$$

From (6.25) and (6.28), we obtain the integral of du extended over all u such that $\Gamma_1 \cap u\Gamma_2 \neq 0$, each u counted $N(\Gamma_1 \cap u\Gamma_2)$ times. We get (up to the sign which is unessential),

$$(6.29) \qquad \int_{\mathfrak{U}} N(\Gamma_1 \cap u\Gamma_2) \, du = 32\pi^6 J_1 J_2,$$

where J_1 and J_2 are the orders of Γ_1 and Γ_2, respectively.

To obtain the mean value of $N(\Gamma_1 \cap u\Gamma_2)$, we need the total measure of $\mathfrak{U}$. Taking for Γ_1 and Γ_2 two straight lines, we know that $J_1 = J_2 = 1$ and $N = 1$; therefore (6.29) gives $\int_{\mathfrak{U}} du = 32\pi^6$. Consequently, the mean value of N is

(6.30) $$\bar{N} = J_1 J_2.$$

For algebraic curves, N is constant and (6.30) gives the classical theorem of Bezout; therefore our result may be considered a generalization of this theorem to meromorphic curves. For the extension to analytic manifolds of P_n see (56).

7. INTEGRAL GEOMETRY IN RIEMANNIAN SPACES

7.1. *Geodesics which intersect a fixed surface.* The methods of the integral geometry can be also applied to Riemannian spaces, mainly to spaces of constant curvature or other spaces which admit a group of transformations into themselves. The case of surfaces is simple and well known (55). Here, we want to consider the case of 3-dimensional spaces.

Let R_3 be a 3-dimensional Riemannian space defined by $ds^2 = g_{ij}\, dx_i\, dx_j$, where the summation convention is adopted; i, j are summed from 1 to 3. Let us introduce the notations,

(7.1) $$F = (g_{ij}x_i'x_j')^{1/2}, \qquad p_i = \frac{\partial F}{\partial x_i'},$$

where $x_i' = dx_i/dt$. As we know, a geodesic of R_3 is determined by a point x_i and a direction x_i', which is equivalent to give x_i, p_i ($i = 1, 2, 3$). The density for sets of geodesics is defined by the following exterior differential form, taken always in absolute value:

(7.2) $$dG = dp_2 \wedge dx_2 \wedge dp_3 \wedge dx_3 + dp_3 \wedge dx_3 \wedge dp_1 \wedge dx_1 + dp_1 \wedge dx_1 \wedge dp_2 \wedge dx_2.$$

The measure of a set of geodesics is the integral of dG extended over the set. The density (7.2) is the second power of the differential invariant $\sum_1^3 dp_i \wedge dx_i$, which constitutes the invariant integral of Poincaré of the dynamics (6, pp. 19 and 78), and it therefore possesses the following two properties of invariance: (1) it is invariant with respect to a change of coordinates in the space; (2) it is invariant under displacements of the elements (x_i, p_i) on the respective geodesic.

To give a geometrical interpretation of dG, let us consider a fixed surface Σ and a set of geodesics which intersect Σ. Let G be such a geodesic and P its intersection point with Σ. In a neighborhood of P we may assume that the equation of Σ is $x_3 = 0$ and that the coordinate system is orthogonal, that is, $ds^2 = g_{11}\,dx_1^2 + g_{22}\,dx_2^2 + g_{33}\,dx_3^2$, and thus $p_i = g_{ii}(dx_i/ds)$. If ν_i represents the cosine of the angle between G and the x_i-coordinate curve at P, we have

$$(7.3)\quad \nu_i = \sqrt{g_{ii}}\frac{dx_i}{ds}, \quad p_i = \sqrt{g_{ii}}\nu_i, \quad dp_i = \sqrt{g_{ii}}\,d\nu_i + \frac{\partial\sqrt{g_{ii}}}{\partial x_h}\nu_i\,dx_h.$$

To determine G according to the second property of invariance of dG, we may choose its intersection point P with Σ. At this point we have $x_3 = 0$, $dx_3 = 0$, and, consequently, (7.2) takes the form

$$(7.4)\qquad dG = dp_1 \wedge dx_1 \wedge dp_2 \wedge dx_2,$$

or, according to (7.3),

$$(7.5)\qquad dG = \sqrt{g_{11}g_{22}}\,d\nu_1 \wedge dx_1 \wedge d\nu_2 \wedge dx_2.$$

On the other hand, to each set of direction cosines ν_1, ν_2, and ν_3 corresponds a point of the unit euclidean sphere and the area element in it has the value (3.15)

$$(7.6)\qquad d\sigma = \frac{d\nu_1 \wedge d\nu_2}{\nu_3}.$$

Hence, we have, in absolute value,

$$(7.7)\qquad dG = |\cos\varphi|\,d\sigma \wedge df,$$

where φ is the angle between the tangent to G and the normal to Σ at P, and $df = \sqrt{g_{11}g_{22}}\,dx_1 \wedge dx_2$ is the element of area Σ at P.

Integrating over all geodesics which intersect Σ, on the left side each geodesic is counted a number of times equal to the number n of intersection points of G and Σ; on the right, the integral of $|\cos\varphi|\,d\sigma$ gives one-half the projection of the unit sphere upon a diametral plane ($= \pi$). Consequently, we get the integral formula

$$(7.8)\qquad \int n\,dG = \pi F,$$

where F is the area of Σ. This formula generalizes (3.28) to Riemannian spaces.

7.2. *Sets of geodesic segments.* Let t be the arc length on the geodesic G. From (7.7) we deduce

$$dG \wedge dt = |\cos \varphi|\, d\sigma \wedge df \wedge dt. \tag{7.9}$$

The product $|\cos \varphi|\, dt$ equals the projection of the arc element dt upon the normal to Σ at P; consequently, $|\cos \varphi|\, df \wedge dt$ equals the element of volume dP of the space at P, and (7.9) can be written in the form,

$$dG \wedge dt = dP \wedge d\sigma. \tag{7.10}$$

An oriented segment S of geodesic is determined either by G, t (G = geodesic which contains S; t = abscissa on G of the origin of S) or by P (= origin of S) and the point of the unit euclidean sphere which gives the direction of S. The two equivalent forms (7.10) may therefore be taken as density for sets of segments of geodesic lines.

For example, let us consider the set of oriented segments S with the origin inside a fixed domain D. The integral of the left of (7.10) gives $2 \int \lambda\, dG$, where λ denotes the length of the arc of G which lies inside D (the factor 2 appears as a consequence that dG means the density for non-oriented geodesic lines). The integral of the right is equal to $4\pi V$, where V is the volume of D. Consequently, we have the following integral formula

$$\int \lambda\, dG = 2\pi V, \tag{7.11}$$

where the integral is extended over all geodesics which intersect D.

7.3. *Some integral formulas for convex bodies in spaces of constant curvature.* Let R_3 now be a 3-dimensional space of constant curvature k. With respect to a system of geodesic polar coordinates, it is known that the element of length can be written in the form

$$ds^2 = d\rho^2 + \frac{\sin^2 \sqrt{k}\rho}{k}\, d\tau, \tag{7.12}$$

where ρ denotes the geodesic distance from a fixed point (origin of coordinates) and $d\tau$ represents the length element of the 2-dimensional unit euclidean sphere. The volume element has the form

$$dP = \frac{\sin^2 \sqrt{k}\rho}{k}\, d\rho \wedge d\sigma, \tag{7.13}$$

where $d\sigma$ denotes the element of area on the unit sphere.

Let P_1, P_2 be two points in R_3 such that there is only one geodesic G which unites them. Let ρ_1, ρ_2 be the abscissas on G of P_1 and P_2. With respect to a system of geodesic polar coordinates with the origin at P_1, the element of volume dP_2 has the form

$$dP_2 = \frac{\sin^2 \sqrt{k}\, |\rho_2 - \rho_1|}{k}\, d\rho_2 \wedge d\sigma. \tag{7.14}$$

By exterior multiplication by dP_1, we have, in consequence of (7.10),

$$dP_1 \wedge dP_2 = \frac{\sin^2 \sqrt{k}\, |\rho_2 - \rho_1|}{k}\, d\rho_1 \wedge d\rho_2 \wedge dG. \tag{7.15}$$

This formula is the work of Haimovici (27).

Let D be a convex domain of volume V (that is, it contains, with each pair of its points, the arc of geodesic, assumed unique, determined by them) and consider all the pairs P_1, P_2 inside D. The integral of the left side of (7.15) is equal to V^2. If λ denotes the length of the arc of G which lies inside D, then by calculating the integral of the right side we have

$$\int_0^\lambda \int_0^\lambda \sin^2 \sqrt{k}\, |\rho_2 - \rho_1|\, d\rho_1\, d\rho_2 = \frac{1}{2}\left(\lambda^2 - \frac{1}{k}\sin^2 \sqrt{k}\lambda\right).$$

Hence, we have the integral formula

$$\frac{1}{k}\int \left(\lambda^2 - \frac{1}{k}\sin^2 \sqrt{k}\lambda\right) dG = 2V^2, \tag{7.16}$$

where the integral is extended over all geodesics which intersect D.

For the elliptic space ($k = 1$), this formula reduces to

$$\int (\lambda^2 - \sin^2 \lambda)\, dG = 2V^2, \tag{7.17}$$

and for the hyperbolic space ($k = -1$),

$$\int (\sinh^2\lambda - \lambda^2)\, dG = 2V^2. \tag{7.18}$$

For the euclidean space ($k = 0$), passing to the limit for $k \to 0$ in (7.16) we get

$$\int \lambda^4\, dG = 6V^2, \tag{7.19}$$

which is a formula of Herglotz [Blaschke (3)].

Formulas of this kind referring to convex figures in the plane or to convex bodies in the euclidean space were first obtained by Crofton (7), considered the creator of the integral geometry. A great deal of them were given successively by several authors: Lebesgue (34), Blaschke (3), Massoti Biggiogero (38–42). Paper (38) contains an extensive bibliography.

The generalization to spaces of constant curvature is less known. However for certain types of formulas, the treatment in elliptic space is more satisfactory than that in euclidean space, owing to the possibility of dualization. Let us consider the following examples.

In the elliptic 3-dimensional space, all geodesics are closed and have the finite length π. The planes have finite area 2π. Since any geodesic intersects a fixed plane in one and only one point, the formula (7.8) gives the measure of the set of all geodesics of the space:

$$\int dG = 2\pi^2. \tag{7.20}$$

Let D be a convex body of area F and volume V and let us consider the set of geodesic segments of length π which intersect D. The integral on the left of (7.10) extended over this set making use of (7.8) for $n = 2$, has the value

$$\int dG\, dt = \pi \int dG = \frac{\pi^2}{2} F, \tag{7.21}$$

and the integral on the right is

$$\int dP \wedge d\sigma = 2\pi V + \int_{P \notin D} \Phi\, dP, \tag{7.22}$$

where Φ denotes the solid angle under which D is seen from P

(P exterior to D). From (7.21) and (7.22), we deduce the integral formula

$$\int_{P \notin D} \Phi \, dP = \tfrac{1}{2}\pi^2 F - 2\pi V. \tag{7.23}$$

Let us now see which formula corresponds to (7.11) by duality. Let M, F be the integral of mean curvature and the area of the boundary of D. For the dual convex body D^* it is known that we have

$$F^* = 4\pi - F, \qquad M^* = M, \qquad V^* = \pi^2 - M - V. \tag{7.24}$$

By duality to each straight line (geodesic) G corresponds another straight line G^* and, hence, if we use (7.24), formula (7.11) gives

$$\int_{G^* \cap D^* = 0} (\pi - \varphi^*) \, dG^* = 2\pi(\pi^2 - M^* - V^*),$$

where φ^* denotes the angle between the two supporting planes of D^* through G^* and the integral is extended over all geodesics G^* exterior to D^*. Taking into account (7.20) and (7.8), and replacing G^* by G, we get the integral formula

$$\int_{G \cap D = 0} \varphi \, DG = 2\pi(M + V) - \tfrac{1}{2}\pi^2 F, \tag{7.25}$$

which has no analogue in the euclidean geometry.

Similarly, as dual of the formula (7.17), we have

$$\int_{G \cap D = 0} (\varphi^2 - \sin^2 \varphi) \, dG = 2(M + V)^2 - \tfrac{1}{2}\pi^3 F, \tag{7.26}$$

where, as in (7.25), φ denotes the angle between the two supporting planes of D through G and the integral is extended over all geodesics which do not intersect D. For the integral geometry in spaces of constant curvature, see Petkantschin (48), and (53), (54), and (59).

8. SUPPLEMENTARY REMARKS AND BIBLIOGRAPHICAL NOTES

8.1. *General integral geometry.* The integral geometry has its origin in the theory of geometrical probabilities [Crofton (13), Deltheil (14), and Herglotz (29), and it was widely developed by Blaschke and his school in a series of papers quoted in Reference (3). The inclusion of the methods and results of the integral geometry within the framework of the theory of homogeneous spaces (as we have done in Section 2) is the work of Weil (73) and (74), and Chern (7). After their work, the measure theory in groups and homogeneous spaces became of fundamental interest in integral geometry. Every new result in that direction can be applied and probably exploited with success to get integral geometric statements; at least, it is sure that the integral geometry constitutes the most abundant source of examples [Nachbin (44) and Helgason (28, Chap. X)].

The inverse problem of finding a general formulation of certain particular formulas of integral geometry (Crofton's formulas) is also an interesting one [Hermann (30) Legrady (36)]. A very simple example follows. We have seen that the kinematic density for the group of motions $\mathfrak{M}$ of the plane is $dK = dP \wedge d\alpha$ (1.11). From the point of view of the homogeneous spaces, dP is the density of the space $\mathfrak{M}/\mathfrak{M}_1$, where $\mathfrak{M}_1$ denotes the group of rotations about a fixed point and $d\alpha$ is the density of $\mathfrak{M}_1$. If we write, symbolically, $dK = d\mathfrak{M}$, $dP = d(\mathfrak{M}/\mathfrak{M}_1)$, $d\alpha = d\mathfrak{M}_1$, the formula (1.11) gives $d\mathfrak{M} = d(\mathfrak{M}/\mathfrak{M}_1) \wedge d\mathfrak{M}_1$, which induces us to ask if it will hold for a general group $\mathfrak{G}$ and its subgroup $\mathfrak{g}$. In this particular example, it is well known that the formula $d\mathfrak{G} = d(\mathfrak{G}/\mathfrak{g}) \wedge d\mathfrak{g}$, in fact, holds for any locally compact topological group $\mathfrak{G}$ and any closed subgroup $\mathfrak{g}$ of $\mathfrak{G}$ [Weil (73, pp. 42–45) and Ambrose (2)].

8.2. *Sets of manifolds.* Some problems of integral geometry may also be presented under the following form. Let V denote a differentiable manifold and F a family of submanifolds in it. First we ask for the existence of a transformation group $\mathfrak{G}$ of V onto itself which transforms the elements of F onto elements of F. Then, if such a group exists, we ask for a measure of sets of varieties of F invariant under $\mathfrak{G}$. We shall give two simple examples.

Examples

1. Let V be the euclidean plane E_2 and F the family of all circles of it. The group $\mathfrak{G}$ is known to be the group of similitudes

$$(8.1)\quad x' = \rho(x\cos\varphi - y\sin\varphi) + a, \qquad y' = \rho(x\sin\varphi + y\cos\varphi) + b,$$

which depends on the 4 parameters a, b, ρ, and φ. This group can be represented by the group of matrices,

$$u = \begin{pmatrix} \rho\cos\varphi & -\rho\sin\varphi & a \\ \rho\sin\varphi & \rho\cos\varphi & b \\ 0 & 0 & 1 \end{pmatrix},$$

and by the method of Section (2.2), we find immediately that the forms of Maurer-Cartan are

$$\omega_1 = \frac{d\rho}{\rho}, \qquad \omega_2 = d\varphi, \qquad \omega_3 = \frac{\cos\varphi}{\rho}\,da + \frac{\sin\varphi}{\rho}\,db,$$

$$\omega_4 = \frac{\sin\varphi}{\rho}\,da + \frac{\cos\varphi}{\rho}\,db.$$

The similitudes which leave invariant a given circle are characterized by a, b, ρ = constants, and, consequently, the system (2.3) is $\omega_1 = 0$, $\omega_3 = 0$, $\omega_4 = 0$. The density for sets of circles (of center a, b and radius ρ) invariant under the group of similitudes results:

$$dC = \frac{da \wedge db \wedge d\rho}{\rho^3}.$$

2. Let V be the real projective plane and F the family of nondegenerate conics in it. Then the group G is the projective group and the density for conics is (61),

$$dC = \frac{da_{00} \wedge da_{01} \wedge da_{02} \wedge da_{11} \wedge da_{12}}{3\Delta^2}$$

where $\Delta = \det(a_{ij})$ and the equation of the conic is assumed to be

$$a_{00}x_0^2 + 2a_{01}xy + a_{11}y^2 + 2a_{02}x + 2a_{12}y + 1 = 0.$$

Other examples of this kind have been given by Stoka (63–68). For sets of degenerate conics, see Luccioni (37).

8.3. *Integral geometry of special groups.* The metric (euclidean and noneuclidean) integral geometry is the best known; however, other cases have also been investigated. The integral geometry of the unimodular affine group of the euclidean space onto itself leads to certain affine invariants for convex bodies (62). The integral geometry of the projective group has been considered by Varga (70) and is pursued in (55); that of the symplectic group has been studied by Legrady (35).

In the last years, Gelfand and his school have largely generalized the ideas of the integral geometry and used them in problems of group representation (21).

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