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ON THE MEASURE OF LINE SEGMENTS ENTIRELY CONTAINED IN A CONVEX BODY

L.A. SANTALÓ

Academia Nacional de Ciencas Exatas, Fisicas y Naturales, Buenos Aires, Argentina

Dedicated to Leopoldo Nachbin with admiration and friendship

Let K be a convex body in the n-dimensional euclidean space $\mathbf{R}^n$. We consider the measure $M_0(l)$, in the sense of the integral geometry (i.e. invariant under the group of translations and rotations of $\mathbf{R}^n$ [6, Chap. 15]), of the set of non-oriented line segments of length l, which are entirely contained in K. This measure is related by (3.4) with the integrals I_m for the power of the chords of K. These relations allow to obtain some inequalities, like (3.6), (3.7) and (3.8) for $M_0(l)$. Next we relate $M_0(l)$ with the function $\Omega(l)$ introduced by Enns and Ehlers [3], and prove a conjecture of these authors about the maximum of the average of the random straight line path through K. Finally, for $n = 2$, $M_0(l)$ is shown to be related by (5.6) with the associated function to K introduced by W. Pohl [5]. Some representation formulas, like (3.9), (3.10) and (5.14) may be of independent interest.

1. Integrals for the Power of the Chords of a Convex Body

Let K be a convex body in the n-dimensional euclidean space $\mathbf{R}^n$. Let dG be the density for lines G in $\mathbf{R}^n$ in the sense of integral geometry [6, Chap. 12] and let σ denote the length of the chord $G \cap K$. The chord power integrals

(1.1) $$I_m = \int_{G\cap K \neq \emptyset} \sigma^m \, dG \qquad (m \geq 0),$$

have been well studied [6, p. 237]. If dP_1, dP_2 denote the elements of volume of $\mathbf{R}^n$ at the points $P_1, P_2 \in K$ and r denotes the distance between P_1 and P_2, the integrals

(1.2) $$J_m = \int_{P_1, P_2 \in K} r^m \, dP_1 \wedge dP_2 \qquad (m \geq -(n-1)),$$

have also been considered and it is known that the relation

$$2I_m = m(m-1)J_{m-n-1} \tag{1.3}$$

holds good for $m > 1$ [6, p. 238].

For the cases $m = 0, 1$ and $m = n+1$ we have the simple formulas

$$I_0 = \tfrac{1}{2}\frac{O_{n-2}}{n-1}F, \qquad I_1 = (\tfrac{1}{2}O_{n-1})V, \qquad I_{n+1} = (\tfrac{1}{2}n(n+1))V^2, \tag{1.4}$$

where F is the surface area of K and V its volume.

We want to calculate I_m for the sphere S_r of radius r in $\mathbf{R}^n$. To this end, recalling that $dG = d\sigma_{n-1} \wedge dO_{n-1}$ [6, (12.39)] where $d\sigma_{n-1}$ is the area element of an hyperplane orthogonal to G at its intersection point with G and dO_{n-1} is the area element of the unit sphere at the end point of the unit vector parallel to G, we can write $dG = \rho^{n-2}\,dO_{n-2} \wedge d\rho \wedge dO_{n-1}$ and therefore we have (ρ being the distance from the center of the sphere to G)

$$\begin{aligned} I_m &= 2^{m-1}O_{n-1}O_{n-2}\int_0^r (r^2-\rho^2)^{m/2}\rho^{n-2}\,d\rho \\ &= 2^{m-2}O_{n-1}O_{n-2}r^{m+n-1}\,\mathrm{B}(\tfrac{1}{2}(n-1), \tfrac{1}{2}(m+2)), \end{aligned} \tag{1.5}$$

where $\mathrm{B}(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the Beta function and O_h means the surface area of the h-dimensional unit sphere, i.e.

$$O_h = \frac{2\pi^{(h+1)/2}}{\Gamma(\frac{1}{2}(h+1))}. \tag{1.6}$$

Therefore we have

$$I_m(S_r) = \frac{2^{m-1}r^{m+n-1}\pi^{n-1/2}m\Gamma(\frac{1}{2}m)}{\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m+n+1))}. \tag{1.7}$$

2. Inequalities of Hadwiger, Carleman and Blaschke for the Chord Integrals I_m

The chord integrals I_m for convex bodies in $\mathbf{R}^n$ satisfy certain inequalities. One of them is due to H. Hadwiger [4]:

(2.1) $$2I_{n-1} \le n\left(\frac{\kappa_{2n-2}}{n\kappa_{n-1}}\right)^{1/n} F V^{1-1/n} \qquad (n > 2),$$

where κ_h denotes the volume of the n-dimensional unit ball, i.e.

(2.2) $$\kappa_h = \frac{O_{h-1}}{h} = \frac{2\pi^{h/2}}{h\Gamma(\frac{1}{2}h)},$$

and F and V denote the surface area and the volume of K respectively.

Taking into account the isoperimetric inequality

(2.3) $$n\kappa_n^{1/n} V^{1-1/n} \le F,$$

inequality (2.1) gives

(2.4) $$4I_{n-1} \le F^2 \qquad (n > 2).$$

In (2.1) and (2.4) the equality sign holds only for the sphere.

In [2] T. Carleman proved that in the plane, $n = 2$, $J_{-1} = \int r^{-1}\, dP_1 \wedge dP_2 = I_2$ has its maximum for the circle (for a given surface area) and pointed out that the same proof may be extended to showing that for convex bodies in $\mathbf{R}^n$, the integrals I_m for $m = 2, 3, \ldots, n$ have a maximum for the sphere for a given volume V. Thus, taking into account (1.7) and (2.2) we have the following set of inequalities:

(2.5)
$$I_m^n \le 2^{mn-m-n+1} \pi^{(n^2/2)-mn/2} n^{m+n-1} (\Gamma(\tfrac{1}{2}n))^{m-1} \left(\frac{\Gamma(\frac{1}{2}m+1)}{\Gamma(\frac{1}{2}(m+n+1))}\right)^n V^{m+n-1},$$

for $m = 2, 3, \ldots, n$.

In [1], W. Blaschke proved that in the plane ($n = 2$) and for a given area F, the integrals I_m ($m \ge 4$) have its minimum for the circle. The proof is also easily extendible to $\mathbf{R}^n$, so that, taking (1.7) and (2.2) into account, we can write the new set of inequalities (for $\mathbf{R}^n$)

(2.6)
$$I_m^n \ge 2^{mn-m-n+1} \pi^{(n^2/2)-mn/2} n^{m+n-1} (\Gamma(\tfrac{1}{2}n))^{m-1} \left(\frac{\Gamma(\frac{1}{2}m+1)}{\Gamma(\frac{1}{2}(m+n+1))}\right)^n V^{m+n-1},$$

for $m \ge n+2$. In (2.5) and (2.6) the equality sign holds only for the sphere.

3. The Measure $M_0(l)$ of the Line Segments of Length l Entirely Contained in a Convex Body K in $\mathbf{R}^n$

A line segment S of given length l in $\mathbf{R}^n$ can be determined either by the line G which contains the segment and the abscissa t of the origin P of S on G, or by P and the point on the unit sphere O_{n-1} given by the direction of S. The kinematic density for sets of line segments of length l (invariant under motions in $\mathbf{R}^n$) is [6, p. 338]:

(3.1)
$$\mathrm{d}S = \mathrm{d}G \wedge \mathrm{d}t = \mathrm{d}P \wedge \mathrm{d}O_{n-1} .$$

Using $\mathrm{d}S = \mathrm{d}G \wedge \mathrm{d}t$ we get that the measure of the set of line segments S entirely contained in K is

(3.2)
$$M_0(l) = \int_{\sigma \geq l} (\sigma - l)\,\mathrm{d}G .$$

If P_1, P_2 are two points of K at a distance l, we have $\mathrm{d}P_1 \wedge \mathrm{d}P_2 = l^{n-1}\,\mathrm{d}O_{n-1} \wedge \mathrm{d}l \wedge \mathrm{d}P_1$ (up to the sign) and therefore, since we consider the measure of non-oriented segments, we have

(3.3)
$$\int_{P_1, P_2 \in K} l^m\,\mathrm{d}P_1 \wedge \mathrm{d}P_2 = 2 \int_0^{\mathrm{Diam}(K)} l^{m+n-1} M_0(l)\,\mathrm{d}l .$$

As a consequence of (1.3) and (3.3) we have

(3.4)
$$I_m = m(m-1)J_{m-n-1} = m(m-1) \int_0^{\mathrm{Diam}(K)} l^{m-2} M_0(l)\,\mathrm{d}l ,$$

which holds for $m \geq 2$. In particular, for $m = 2$ we have

(3.5)
$$I_2 = 2 \int_0^{\mathrm{Diam}(K)} M_0(l)\,\mathrm{d}l ,$$

and the first inequality (2.5) gives

(3.6) $$\int_0^{\mathrm{Diam}(K)} M_0(l)\,\mathrm{d}l \leq \frac{2^{1-1/n}\pi^{(n/2)-1}n^{1+1/n}(\Gamma(\frac{1}{2}n))^{1/n}V^{(n+1)/n}}{(n+1)\Gamma(\frac{1}{2}(n+1))},$$

where the equality sign holds only for the sphere.

For instance, for convex sets K in the plane, $n = 2$, we have

(3.7) $$\int_0^{\mathrm{Diam}(K)} M_0(l)\,\mathrm{d}l \leq \frac{8}{3\sqrt{\pi}} F^{3/2},$$

where F is the surface area of K.

Taking into account the isoperimetric inequality $4\pi F \leq L^2$ and the inequality of Bieberbach $F \leq \frac{1}{4}\pi D^2$, where $D = \mathrm{diam}(K)$, we get the following inequalities (for convex sets in the plane):

(3.8) $$\int_0^D M_0(l)\,\mathrm{d}l \leq \frac{L^3}{3\pi^2}, \qquad \int_0^D M_0(l)\,\mathrm{d}l \leq \tfrac{1}{3}\pi D^3,$$

with the equality sign always only for the circle.

From (3.4) we deduce that for every polynomial function of the form $f = a_2\sigma^2 + \cdots + a_h\sigma^h$ we have

(3.9) $$\int_{G\cap K\neq\emptyset} f(\sigma)\,\mathrm{d}G = \int_0^D f''(\sigma)M_0(\sigma)\,\mathrm{d}\sigma.$$

By Weierstrass approximation theorem, this equality holds for every function $f(\sigma)$ having continuous derivatives $f''(\sigma)$ with the conditions $f(0) = f'(0) = 0$.

Integrating by parts the right side of (3.9) we have the following relationship

(3.10) $$\int_{G\cap K\neq\emptyset} f(\sigma)\,\mathrm{d}G = -\int_0^D f'(\sigma)M_0'(\sigma)\,\mathrm{d}\sigma,$$

for every function $f(\sigma)$ having continuous derivative $f'(\sigma)$ and satisfying the condition $f(0) = f'(0) = 0$.

4. The Invariants $\Omega(l)$ of Enns–Ehlers

Denote by $K(l, \omega)$ the translate of K by a distance l in the direction ω. Enns and Ehlers [3] define $\Omega(l)$ to be the volume of $K \cap K(l, \omega)$ uniformly averaged over all directions and normalized such that $\Omega(0) = 1$. If σ denotes the length of the chord $G \cap K$, the volume of $K \cap K(l, \omega)$ is precisely $\int_{\sigma \geqslant l} (\sigma - l)\, \mathrm{d}\sigma_{n-1}$, where $\mathrm{d}\sigma_{n-1}$ denotes the area element on the hyperplane orthogonal to the line G which has the direction ω. Therefore, since $\mathrm{d}G = \mathrm{d}\sigma_{n-1} \wedge \mathrm{d}O_{n-1}$, where $\mathrm{d}O_{n-1}$ denotes the area element on the unit $(n-1)$-sphere corresponding to the direction ω, we have

$$(4.1) \qquad \Omega(l) = \frac{2}{O_{n-1}V} \int\limits_{\sigma \geqslant l} (\sigma - l)\, \mathrm{d}\sigma_{n-1} \wedge \mathrm{d}O_{n-1} = \frac{2}{O_{n-1}V} \int\limits_{\sigma \geqslant l} (\sigma - l)\, \mathrm{d}G$$

and thus, according to (3.2),

$$(4.2) \qquad \Omega(l) = \frac{2}{O_{n-1}V} M_0(l).$$

Therefore, (3.4) gives

$$(4.3) \qquad I_m = \tfrac{1}{2} m(m-1) O_{n-1} V \int_0^D l^{m-2} \Omega(l)\, \mathrm{d}l.$$

For instance, if $m = n + 1$, taking (1.4) into account, we have

$$(4.4) \qquad \int_0^D l^{n-1} \Omega(l)\, \mathrm{d}l = \frac{V}{O_{n-1}},$$

according to a result of Enns and Ehlers [3, (8)].

If a 'random secant' is defined by a point in the interior of K and by a direction (the point and direction have independent uniform distribution), the k-th moment of a random secant is (using (3.1))

$$E(\sigma^k) = \frac{2}{O_{n-1}V}\int \sigma^k \, \mathrm{d}P \wedge \mathrm{d}O_{n-1} = \frac{2}{O_{n-1}V}\int \sigma^{k+1}\,\mathrm{d}G \tag{4.5}$$

$$= \frac{2k(k+1)}{O_{n-1}V}\int_0^D \sigma^{k-1} M_0(\sigma)\,\mathrm{d}\sigma .$$

Thus, according to (3.4) we have

$$E(\sigma^k) = \frac{2}{O_{n-1}V} I_{k+1},$$

and the inequalities (2.5) give

$$E(\sigma^k) \leq \frac{2^{k-k/n} n^{(k/n)+1} (\Gamma(\frac{1}{2}n))^{(k+n)/n} \Gamma(\frac{1}{2}(k+3))}{\pi^{(k+1)/2}\Gamma(\frac{1}{2}(k+n+2))} V^{k/n}, \tag{4.6}$$

which holds for $k = 1, 2, \ldots, n-1$ and the equality sign holds only for the sphere. For the sphere of radius r we have $V = (2\pi^{n/2}/n\Gamma(\frac{1}{2}n))r^n$ and therefore

$$E(\sigma^k) = \frac{2^k n \Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(k+3))}{\pi^{1/2}\Gamma(\frac{1}{2}(k+n+2))} r^k, \tag{4.7}$$

as is well-known (Enns–Ehlers [3]).

In particular (4.6) implies that of all n-dimensional convex bodies K of volume V, $E(\sigma)$ is maximized for the n-sphere. This proves a conjecture of Enns and Ehlers [3].

The inequalitiies (2.6) can be written

$$E(\sigma^k) \geq \frac{2^{k-k/n} n^{1+k/n} (\Gamma(\frac{1}{2}n))^{1+k/n} \Gamma(\frac{1}{2}(k+3))}{\pi^{(k+1)/2}\Gamma(\frac{1}{2}(k+n+2))} V^{k/n}, \tag{4.8}$$

valid for $k = n+1, n+2, \ldots$. The equality sign holds only for the sphere.

For the plane, $n = 2$, if F denotes the area of K, we have

$$E(\sigma) \leq \frac{8F^{1/2}}{3\pi^{3/2}}, \qquad E(\sigma^2) = \frac{3}{\pi} F, \tag{4.9}$$

and therefore, of all the convex sets of area F, the variance

$$E(\sigma^2) - (E(\sigma))^2 \geqslant \frac{27\pi^2 - 64}{9\pi^3} F$$

is minimized for the circle (as conjectured by Enns–Ehlers [3]). The conjecture that the variance is also minimized for the sphere if $n > 2$ remains open.

5. The Associated Functions $A(\sigma)$ of W. Pohl

In this section we consider only the case of the plane, $n = 2$. In each line G we choose a point $X(x, y)$ and the unit vector $e(\cos\theta, \sin\theta)$ corresponding to its direction. Consider the differential form $\omega = \langle \mathrm{d}X, e\rangle = \cos\theta \cdot \mathrm{d}x + \sin\theta \cdot \mathrm{d}y$. Then we have $\mathrm{d}\omega = -\sin\theta \cdot \mathrm{d}\theta \wedge \mathrm{d}x + \cos\theta \cdot \mathrm{d}\theta \wedge \mathrm{d}y = \mathrm{d}G$ (according to [6, (3.11)]). W. Pohl [5] defines the associated function $A(\sigma)$ to the convex curve ∂K by

$$A(\sigma) = \int_{\partial K} \omega = \int_{\partial K} \cos\alpha \,\mathrm{d}s\,, \tag{5.1}$$

where the integral of ω extends to the non-oriented lines (X, e), $X \in \partial K$, that determine on the convex set K a chord of length σ and in the last integral α denotes the angle between the tangent to ∂K and G at the point X corresponding to the element of the arc $\mathrm{d}s$.

A simple geometric description of $A(\sigma)$, at least for small values of σ, is the following [5]: Let ∂K_0 be the curve envelope of the chords of K of length σ. Then $A(\sigma)$ is length of ∂K_0. For instance, for a circle of diameter D we have

$$A(\sigma) = \pi(D^2 - \sigma^2)^{1/2}\,. \tag{5.2}$$

Notice that our $A(\sigma)$ is one half of that of Pohl, which considers oriented lines.

Let $M_1(l)$ be the measure of the set of non-oriented line segments of length l such that one end point is inside K and one outside K. Then we have [5]

(5.3) $$M_1(l) = 2\int_0^l A(\sigma)\,\mathrm{d}\sigma\,.$$

On the other side, using the kinematic density $\mathrm{d}S = \mathrm{d}G \wedge \mathrm{d}t$, we have

(5.4) $$M_1(l) = 2\int_{\sigma \geq l} l\,\mathrm{d}G + 2\int_{\sigma \leq l} \sigma\,\mathrm{d}G$$

and by virtue of (3.2) we get

(5.5) $$M_0 + \tfrac{1}{2}M_1 = \pi F\,,$$

where F is the surface area of K. From (5.3) and (5.5) we have

(5.6) $$M_0 = \pi F - \int_0^l A(\sigma)\,\mathrm{d}\sigma\,,$$

and

(5.7) $$A(\sigma) = -M_0'(\sigma)\,.$$

The relation (5.6) can be applied to compute the measure $M_0(l)$ of non-oriented line segments of length $l \leq D$ entirely contained in a circle of diameter D. Namely, from (5.2) we have

(5.8) $$M_0(l) = \pi F - \pi\int_0^l (D^2 - \sigma^2)^{1/2}\,\mathrm{d}\sigma$$

$$= \tfrac{1}{4}\pi\left(\pi D^2 - 2l(D^2 - l^2)^{1/2} - 2D^2 \arcsin\left(\frac{l}{D}\right)\right),$$

as is well known [6, p. 90].

Integrating by parts in (3.4) and taking into account (5.7), we get (for convex sets in the plane and $m \geq 1$)

$$I_m = m \int_0^D \sigma^{m-1} A(\sigma)\, \mathrm{d}\sigma , \tag{5.9}$$

where D is the diameter of K. This expression for the chord integrals I_m (for convex sets on the plane) is due to Pohl [5]. For $m = 1$ we have

$$\int_0^D A(\sigma)\, \mathrm{d}\sigma = \pi F . \tag{5.10}$$

For $m = 2$, according to (2.5) we get the inequality

$$\int_0^D \sigma A(\sigma)\, \mathrm{d}\sigma \leqslant \frac{8}{3\sqrt{\pi}} F^{3/2} . \tag{5.11}$$

For $m = 3$ we have

$$\int_0^D \sigma^2 A(\sigma)\, \mathrm{d}\sigma = F^2 , \tag{5.12}$$

and for $m > 3$,

$$\int_0^D \sigma^{m-1} A(\sigma)\, \mathrm{d}\sigma \geqslant \frac{2^{m-1} \pi^{1-m/2} \Gamma(\frac{1}{2} m)}{\Gamma(\frac{1}{2}(m+3))} F^{(m+1)/2} . \tag{5.13}$$

In (5.11) and (5.13) the equality sign holds only for the circle.
From (3.10) and (5.7) we have

$$\int_{G \cap K \neq \emptyset} f(\sigma)\, \mathrm{d}G = \int_0^D f'(\sigma) A(\sigma)\, \mathrm{d}\sigma , \tag{5.14}$$

which holds for every function $f(\sigma)$ having a continuous derivative $f'(\sigma)$ and satisfying the conditions $f(0) = f'(0) = 0$.

The relation between the invariant $\Omega(\sigma)$ of Enns–Ehlers and the associated function $A(\sigma)$ of Pohl, according to (4.2) and (5.6) is

$$A(\sigma) = -\pi F \Omega'(\sigma)\,. \tag{5.15}$$

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