3.1

# Mean Values and Curvatures

L. A. SANTALÓ

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We divide this exposition into two parts. Section 3.1.1 refers to the mean value of the Euler–Poincaré characteristic of the intersection of two convex hypersurfaces in $E_4$. Section 3.1.2 deals with the definition of $q$th total absolute curvatures of a compact $n$-dimensional variety imbedded in Euclidean space of $n+N$ dimensions, extending some results given in [10].

## 3.1.1 ON CONVEX BODIES IN $E_4$

### *Introduction*

Let $K$ be a convex body in 4-dimensional Euclidean space $E_4$ and let $W_i$ $(i = 0, 1, 2, 3, 4)$ be its Minkowski *Quermass integral* (see for instance Bonnesen–Fenchel [1]). Recall that

$$(1)\qquad \begin{cases} W_0 = V = \text{volume of } K, \\ 4W_1 = F = \text{area of } \partial K, \\ W_4 = \pi^2/2 \end{cases}$$

and, if $K$ has sufficiently smooth boundary, we have also

$$(2)\qquad \begin{cases} 4W_2 = M_1 = \text{first mean curvature} = \dfrac{1}{3}\displaystyle\int_{\partial K}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)d\sigma, \\ 4W_3 = M_2 = \text{second mean curvature} = \dfrac{1}{3}\displaystyle\int_{\partial K}\left(\frac{1}{R_1R_2}+\frac{1}{R_1R_3}+\frac{1}{R_2R_3}\right)d\sigma, \end{cases}$$

where $R_i$ are the principal radii of curvature and $d\sigma$ is the element of area of $\partial K$.

For instance, if $K =$ sphere of radius $r$, we have

$$V = \tfrac{1}{2}\pi^2 r^4, \quad F = 2\pi^2 r^3, \quad M_1 = 2\pi^2 r^2, \quad M_2 = 2\pi^2 r. \tag{3}$$

We will use throughout the invariants $V, F, M_1, M_2$ because they have a more geometrical meaning; however, we do not assume smoothness of $\partial K$, so that as definition of $M_1, M_2$ we take $M_1 = 4W_2$, $M_2 = 4W_3$.

The invariants $V, F, M_1, M_2$ are not independent. They are related by certain inequalities which may be written in the following symmetrical form (following Hadwiger [6]).

$$W_\alpha^{\beta-\gamma} W_\beta^{\gamma-\alpha} W_\gamma^{\alpha-\beta} \geq 1, \quad 0 \leq \alpha < \beta < \gamma \leq 4. \tag{4}$$

In explicit form and using the invariants $V, F, M_1, M_2$ the inequalities (4) give the following non-independent inequalities

$$\left\{\begin{array}{lll} F^2 \geq 4VM_1, & F^3 \geq 16V^2M_2, & F^4 \geq 128\pi^2 V^3, \\ M_1^3 \geq 4VM_2^2, & M_1^2 \geq 2\pi^2 V, & M_2^4 \geq 32\pi^6 V, \\ M_1^2 \geq FM_2, & M_1^3 \geq 2\pi^2 F^2, & M_2^3 \geq 4\pi^4 F, \\ M_2^2 \geq 2\pi^2 M_1. & & \end{array}\right. \tag{5}$$

We will represent throughout the paper by $O_i$ the volume of the $i$-dimensional unit sphere, that is

$$O_i = \frac{2\pi^{(i+1)/2}}{\Gamma(\frac{1}{2}[i+1])}; \tag{6}$$

for instance,

$$O_0 = 2, \quad O_1 = 2\pi, \quad O_2 = 4\pi, \quad O_3 = 2\pi^2, \quad O_4 = \tfrac{8}{3}\pi^2, \quad O_5 = \pi^3. \tag{7}$$

*Mean value of* $\chi(\partial K \cap g\,\partial K)$

Let $G$ be the group of isometries of $E_4$. For any $g \in G$ we represent by $g\,\partial K$ the image of $\partial K$ under the isometry $g$. Let $dg$ denote the invariant volume element of $G$ (= kinematic density for $E_4$). Assume the convex body $K$ fixed and consider the intersections $\partial K \cap g\,\partial K$, $g \in G$. Then, Federer [5] and Chern [2] have proved the following integral formula

$$\int_G \chi(\partial K \cap g\,\partial K)\, dg = 64\pi^3 F M_2, \tag{8}$$

where $\chi(\partial K \cap g\,\partial K)$ denotes the Euler–Poincaré characteristic of the surface $\partial K \cap g\,\partial K$.

On the other hand, the so-called fundamental kinematic formula of integral geometry gives

$$\int_{K \cap gK \neq \emptyset} dg = 8\pi^2 (4\pi^2 V + 2FM_2 + \tfrac{3}{2} M_1^2). \tag{9}$$

Therefore the expected value of $\chi(\partial K \cap g\, \partial K)$ is

$$\mathbf{E}(\chi(\partial K \cap g\, \partial K)) = \frac{8FM_2}{4\pi^2 V + 2FM_2 + \tfrac{3}{2} M_1^2}. \tag{10}$$

Notice that, $K$ being convex, the intersections $\partial K \cap g\, \partial K$ are closed orientable surfaces. Thus the possible values of $\chi$ are either $\chi = 2, 4, 6, \ldots$ or $\chi = 0, -2, -4, -5, \ldots$. If $K$ is an Euclidean sphere, obviously we have $\mathbf{E}(\chi) = 2$.

***Conjecture.*** *For all convex sets $K$ of $E_4$ the inequality*

$$\mathbf{E}(\chi(\partial K \cap g\, \partial K)) \leqslant 2 \tag{11}$$

*holds good, with equality for the Euclidean sphere.*†

Putting

$$\Delta = 8\pi^2 V + 3M_1^2 - 4FM_2 \tag{12}$$

the conjecture is equivalent to $\Delta \geqslant 0$. For the Euclidean sphere, according to equations (3) we have $\Delta = 0$.

In support of this conjecture we will prove it for rectangular parallelepipeds. Let $a, b, c, d$ be the sides of a rectangular parallelepiped in $E_4$ and assume

$$a \leqslant b \leqslant c \leqslant d. \tag{13}$$

It is known that (Hadwiger [6])

$$V = abcd, \quad F = 2(abc + abd + acd + bcd),$$

$$M_1 = \tfrac{2}{3}\pi(ab + ac + ad + bc + bd + cd), \quad M_2 = \tfrac{4}{3}\pi(a + b + c + d).$$

† H. Hadwiger (personal communication to the author) has shown that the conjecture is not true. The counter-example is a 4-dimensional right cylinder with a 3-dimensional solid unit sphere as section and altitude equal to 1

$$(V = 4\pi/3,\ F = 20\pi/3,\ M_1 = (4/3)\pi(\pi + 2),\ M_2 = 20\pi/3).$$

Another counter-example is the 3-dimensional solid sphere considered as a flattened convex body of $E_4$ ($V = 0$, $F = 8\pi/3$, $M_1 = 4\pi^2/3$, $M_2 = 16\pi/3$, assuming the radius $r = 1$).

With these values we verify the identity

$$\begin{aligned}\frac{3}{4\pi}\Delta = (4-\pi)\,[a^2c^2+a^2(c-b)^2+b^2(c-a)^2 \\ +a^2(d-b)^2+c^2(d-a)^2+b^2(d-c)^2+c^2(d-b)^2] \\ +(18\pi-56)\,abcd+(4\pi-12)\,(a^2b^2+a^2c^2+b^2c^2) \\ +(8-2\pi)\,[(b-a)\,acd+(c-b)\,abd+(d-c)\,acb] \\ +(4-\pi)\,d^2[(2A^2-B^2)\,(a^2+b^2)+(Ac-Ba)^2+(Ac-Bb)^2],\end{aligned}$$

where $A^2=(3\pi-8)/(8-2\pi)$, $B^2=(8-2\pi)/(3\pi-8)$.

Since all terms are positive, we have $\Delta>0$.

For an ellipsoid of revolution whose semiaxes are $a, a, a, \lambda a$ we have (Hadwiger [6])

$$(14)\qquad \begin{cases} V=(\frac{1}{2}\pi^2)\,\lambda a^4, \quad F=2\pi^2\lambda^2a^3F(\frac{5}{2},\frac{1}{2},2;\,1-\lambda^2),\\ M_1=2\pi^2\lambda^2a^2F(\frac{5}{2},1,2;\,1-\lambda^2),\\ M_2=2\pi^2\lambda^4aF(\frac{5}{2},\frac{3}{2},2;\,1-\lambda^2),\end{cases}$$

where $F$ denotes the hypergeometric function. In this case the conjecture becomes

$$(15)\qquad 1+3\lambda^6F_1^2-4\lambda^6F_{\frac{1}{2}}F_{\frac{3}{2}}\geqslant 0,$$

where

$$F_{\frac{1}{2}}=F(\tfrac{5}{2},\tfrac{1}{2},2;\,1-\lambda^2),$$
$$F_1=F(\tfrac{5}{2},1,2;\,1-\lambda^2),$$
$$F_{\frac{3}{2}}=F(\tfrac{5}{2},\tfrac{3}{2},2;\,1-\lambda^2).$$

I do not know if the inequality (15) holds for all values of $\lambda$.

## 3.1.2 TOTAL ABSOLUTE CURVATURES OF COMPACT MANIFOLDS IMMERSED IN EUCLIDEAN SPACE

### *Introduction*

In this section we extend and complete the contents of [10]. We shall first state some known formulae which will be used in the sequel.

Let $L_h$ be an $h$-dimensional linear subspace in the $(n+N)$-dimensional Euclidean space $E_{n+N}$. We will call it, simply, an $h$-space. Let $L_h(O)$ be an $h$-space in $E_{n+N}$ through a fixed point $O$. The set of all oriented $L_h(O)$ constitute the Grassman manifold $G_{h,n+N-h}$. We shall represent by $dL_h(O)$ the element of volume of $G_{h,n+N-h}$, which is the same thing as the density for oriented $h$-spaces through $O$. The expression of $dL_h(O)$ is well known, but we will recall it briefly for completeness (see [9], [2]).

Let $(O; e_1, e_2, \ldots, e_{n+N})$ be an orthonormal frame in $E_{n+N}$ of origin $O$. In the space of all orthonormal frames of origin $O$ we define the differential forms

$$\omega_{im} = -\omega_{mi} = e_m de_i. \tag{16}$$

Assuming $L_h(O)$ spanned by the unit vectors $e_1, e_2, \ldots, e_h$, then

$$dL_h(O) = \Lambda\omega_{im}, \tag{17}$$

where the right-hand side is the exterior product of the forms $\omega_{im}$ over the range of indices

$$i = 1, 2, \ldots, h; \quad m = h+1, h+2, \ldots, n+N.$$

The $(n+N-h)$-space $L_{n+N-h}(O)$ orthogonal to $L_h(O)$ is spanned by $e_{h+1}, \ldots, e_{n+N}$ and equations (2) give the duality

$$dL_h(O) = dL_{n+N-h}(O). \tag{18}$$

The measure of the set of all oriented $L_h(O)$ (= volume of the Grassman manifold $G_{h,n+N-h}$) may be computed directly from equations (2) (see [9]), or applying the result that it is the quotient space

$$SO(n+N)/SO(h) \times SO(n+N-h)$$

(see [2]). The result is

$$\int_{G_{h,n+N-h}} dL_h(O) = \frac{O_{n+N-1}\, O_{n+N-2} \cdots O_{n+N-h}}{O_1\, O_2 \cdots O_{h-1}} \tag{19}$$

$$= \frac{O_h\, O_{h+1} \cdots O_{n+N-1}}{O_1\, O_2 \cdots O_{n+N-h-1}},$$

where $O_i$ is the area of the $i$-dimensional unit sphere (equation (6)).

Another known integral formula which we will use is the following.

Consider the unit sphere $\Sigma_{n+N-1}$ of dimension $n+N-1$ of centre $O$. Let $V^s$ be an $s$-dimensional variety in $\Sigma_{n+N-1}$. Let $\mu_{s+h-n-N}(V^s \cap L_h)$ be the $(s+h-n-N)$-dimensional measure of the variety $V^s \cap L_h(O)$ of dimension $s+h-(n+N)$ and let $\mu_s(V^s)$ be the $s$-dimensional measure of $V^s$ (all these measures considered as measures of subvarieties of the Euclidean space $E_{n+N}$). Then

$$\int_{G_{h,n+N-h}} \mu_{s+h-n-N}(V^s \cap L_h(O))\, dL_h(O)$$

$$= \frac{O_{n+N-h}\, O_{n+N-h+1} \cdots O_{n+N-1}\, O_{n+s-n-N}}{O_1\, O_2 \cdots O_{h-1}\, O_s} \mu_s(V^s). \tag{20}$$

Note that this formula assumes the $h$-spaces $L_h$ oriented (see [8]). In particular, if $s = 1$ and $h = n+N-1$, that is, for a curve $V^1$ of length $U$, we have

$$\int_{G_{n+N-1,1}} \nu dL_{n+N-1}(O) = \frac{2O_{n+N-1}}{O_1} U, \tag{21}$$

where $\nu$ is the number of points of the intersection $V^1 \cap L_{n+N-1}(O)$.

*Definitions*

Let $X^n$ be a compact $n$-dimensional differentiable manifold (without boundary) of class $C^\infty$ in $E_{n+N}$. To each point $p \in X^n$ we attach the $\rho$-space $T^{(q)}(p)$ spanned by the vectors

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}; \quad \frac{\partial^2}{\partial x_1^2}, \dots, \frac{\partial^2}{\partial x_n^2}; \quad \dots; \quad \frac{\partial^q}{\partial x_1^q}, \dots, \frac{\partial^q}{\partial x_n^q}, \tag{22}$$

which we will call the $q$th tangent fibre over $p$. Its dimension is

$$\rho(n,q) = \sum_{i=1}^{q} \binom{n+i-1}{i}. \tag{23}$$

Assuming

$$1 \leq r \leq n+N-1, \quad \rho \leq n+N-1, \tag{24}$$

we define the $r$th total absolute curvature of order $q$ of $X^n$ as follows.

(*a*) *Case* $1 \leq r \leq \rho$. Let $O$ be a fixed point of $E_{n+N}$ and consider an $(n+N-r)$-space $L_{n+N-r}(O)$. Let $\Gamma_r$ be the set of all $r$-spaces $L_r$ of $E_{n+N}$ which are contained in some of the fibres $T^{(q)}(p)$, $p \in X^n$, pass through $p$, and are orthogonal to $L_{n+N-r}(O)$. The intersection $\Gamma_r \cap L_{n+N-r}(O)$ will be a compact variety in $L_{n+N-r}(O)$ whose dimension $\delta$ we shall compute in the next section. Let $\mu(\Gamma_r \cap L_{n+N-r}(O))$ be the measure of this variety as subvariety of the Euclidean space $L_{n+N-r}(O)$; if $\delta = 0$, then $\mu$ means the number of intersection points of $\Gamma_r$ and $L_{n+N-r}(O)$.

Then we define the $r$th total absolute curvature of order $q$ of $X^n \subset E_{n+N}$ as the mean value of the measures $\mu$ for all $L_{n+N-r}(O)$, that is, according to equality (19)

$$K_{r,N}^{(q)}(X^n) = \frac{O_1 O_2 \dots O_{n+N-r-1}}{O_r O_{r+1} \dots O_{n+N-1}} \int_{G_{n+N-r,r}} \mu(\Gamma_r \cap L_{n+N-r}(O))\, dL_{n+N-r}(O). \tag{25}$$

The coefficient of the right-hand side may be replaced by

$$O_1 O_2 \dots O_{r-1} / O_{n+N-r} \dots O_{n+N-1}.$$

(*b*) *Case* $\rho \leqslant r \leqslant n+N-1$. Instead of the set of $L_r$ which *are contained* in some $T^{(q)}(p)$ we consider now the set of $L_r$ which *contain* some $T^{(q)}(p)$, $p \in X^n$, and are orthogonal to $L_{n+N-r}(O)$. As before we represent this set by $\Gamma_r$ and the $r$th total absolute curvature of order $q$ of $X^n \subset E_{n+N}$ is defined by the same mean value (25).

*Properties*

We proceed now to compute the dimension of $\Gamma_r \cap L_{n+N-r}(O)$.

(*a*) *Case* $1 \leqslant r \leqslant \rho$. The set of all $L_r \subset E_{n+N}$ is the Grassman manifold $G_{r+1,n+N-r}$ whose dimension is $(r+1)(n+N-r)$. The set of all $L_r$ which are contained in $T^{(q)}(p)$ and pass through $p$ is the Grassman manifold $G_{r,\rho-r}$ of dimension $r(\rho-r)$; therefore the set of all $L_r$ which are contained in some $T^{(q)}(p)$, $p \in X^n$, has dimension $r(\rho-r)+n$. On the other hand, the set of all $L_r \subset E_{n+N}$ which are orthogonal to $L_{n+N-r}(O)$ has dimension $n+N-r$. Consequently, the intersection of both sets, as sets of points of $G_{r+1,n+N-r}$, has dimension

$$r(\rho-r)+n+n+N-r-(r+1)(n+N-r) = r\rho+n-r(n+N).$$

Since to each $L_r$ orthogonal to $L_{n+N-r}(O)$ corresponds one and only one intersection point with this linear space, the preceding dimension coincides with the dimension $\delta$ of $\Gamma_r \cap L_{n+N-r}$, that is,

$$\delta = \dim(\Gamma_r \cap L_{n+N-r}(O)) = r\rho+n-r(n+N).$$

Hence, in order that $K_{r,N}^{(q)}(X^n) \neq 0$, it is necessary and sufficient that

$$r\rho+n \geqslant r(n+N). \tag{26}$$

(*b*) *Case* $\rho \leqslant r \leqslant n+N-1$. The set of all $L_r \subset E_{n+N}$ which contain a fixed $L_\rho$, constitute the Grassman manifold $G_{r+\rho,n+N-r}$ and therefore the dimension of the set of all $L_r$ which contain some $T^{(q)}(p)$, $p \in X^n$, is $(r-\rho)(n+N-r)+n$. The remaining dimensions are the same as in the case (*a*), so that the dimension of the set of all $L_r$ which contain some $T^{(q)}(p)$, $p \in X^n$, and are orthogonal to $L_{n+N-r}(O)$, is

$$(r-\rho)(n+N-r)+n+n+N-r-(r+1)(n+N-r) = r\rho+n-\rho(n+N),$$

that is,

$$\delta = \dim(\Gamma_r \cap L_{n+N-r}(O)) = \rho r+n-\rho(n+N).$$

In order that $K_{r,N}^{(q)}(X^n) \neq 0$, it is necessary and sufficient that

$$\rho r+n \geqslant \rho(n+N). \tag{27}$$

Of course, to inequalities (26) and (27) we must add the relations (24).

The most interesting cases correspond to $\delta = 0$, for which the measure $\mu$ in equation (25) is a positive integer and the total absolute curvature is invariant under similitudes. In this case the set of points $p \in X^n$ for which $L_r$ contains or is contained in $T^{(q)}(p)$ can be divided according to the index of $p$, and we get different curvatures in the style of those defined by Kuiper for the case $q = 1$, $r = n+N-1$ [7]. We will not go into details here.

*Examples*

(1) Curves, $n = 1$. For $n = 1$ the condition (26) is

$$1 \geq r + r(N-\rho)$$

and since $\rho \leq N$ the only possibility is $\rho = N$, $r = 1$, which gives $\delta = 0$. The corresponding curvature $K_{1,N}^{(N)}(X^1)$ is

$$K_{1,N}^{(N)}(X^1) = \frac{1}{O_N} \int_{G_{N,1}} \nu_1 \, dL_N(O), \tag{28}$$

where $\nu_1$ is the number of lines in $E_{n+N}$ orthogonal to $L_N(O)$ which are contained in some $N$th tangent fibre of the curve $X^1$. Notice that $G_{N,1}$ is the unit sphere $\Sigma_N$ and $dL_N(O)$ is the element of area of this sphere in consequence of the duality (18). If $e_1, e_2, \ldots, e_{N+1}$ are the principal normals of $X^1$ then the formula (21) says that the right-hand side of equation (28) is equal to the length of the spherical curve $e_{N+1}(s)$ ($s$ = arc length of $X^1$) up to the factor $1/\pi$. That is, if $\kappa_N$ is the $N$th curvature of $X^1$ (see, for instance, Eisenhart [4], p. 107) we have

$$K_{1,N}^{(N)}(X^1) = \frac{1}{\pi} \int_{X^1} |\kappa_N| \, ds. \tag{29}$$

For the case of curves in $E_3$, $N = 2$, $\kappa_N$ is the torsion of the curve and $K_{1,N}^{(2)}$ is up to the factor $\pi^{-1}$, the *absolute total torsion* of $X^1$.

The condition (27) gives $1 \geq \rho + \rho(N-r)$ and since $r \leq N$, this condition implies $\rho = 1$, $r = N$. We have the curvature

$$K_{N,N}^{(1)}(X^1) = \frac{1}{O_N} \int_{G_{1,N}} \nu_N \, dL_1(O), \tag{30}$$

where $\nu_N$ is the number of hyperplanes $L_N$ of $E_{N+1}$ orthogonal to $L_1(O)$ which contain some tangent line of $X^1$. The same formula (21) gives now that the right-hand side of equation (30) is equal to the length of the curve $e_1(s)$ (= spherical tangential image of $X^1$), up to the factor $1/\pi$. Therefore, if $\kappa_1$ is the first curvature of $X^1$, equation (30) becomes

$$K_{N,N}^{(1)}(X^1) = \frac{1}{\pi} \int_{X^1} |\kappa_1| \, ds. \tag{31}$$

Notice that for each direction $L_1(O)$ there are at least two hyperplanes orthogonal to $L_1(O)$ which contain a tangent line of $X^1$ (the hyperplanes which separate the hyperplanes which have a common point with $X^1$ from those which do not). Therefore the mean value $K_{N,N}^{(1)}$ is $\geq 2$ and equation (31) gives the classical Fenchel inequality

$$\int_{X^1} |\kappa_1|\, ds \geq 2\pi. \tag{32}$$

If the curve $X^1$ has at least four hyperplanes orthogonal to an arbitrary direction $L_1(O)$ which contain a tangent line of $X^1$ (as it happens for instance for knotted curves in $E_3$), the mean value $K_{N,N}^{(1)}(X)$ will be $\geq 4$, and we have the Fary inequality

$$\int_{X^1} |\kappa_1|\, ds \geq 4\pi. \tag{33}$$

(2) Surfaces, $n = 2$.

(*i*) *Total absolute curvatures of order* 1. We have $n = 2$, $\rho = 2$ and condition (26) becomes $2 \geq rN$. Therefore the possible cases are $r = 1$, $N = 1$; $r = 2$, $N = 1$ and $r = 1$, $N = 2$. For $2 \leq r \leq N+1$, condition (27) gives $r \geq N+1$ and therefore the only possible case is $r = N+1$.

(*a*) *Case* $r = 1$, $N = 1$. Surfaces in $E_3$. Taking into account that $G_{2,1}$ is the unit sphere $\Sigma_2$, the curvature (25) is

$$K_{1,1}^{(1)}(X^2) = \frac{1}{4\pi} \int_{\Sigma_2} \lambda\, dL_2(O), \tag{34}$$

where $\lambda$ is the length of the curve in the plane $L_2(O)$ generated by the intersections of $L_2(O)$ with the lines of $E_3$ which are tangent to $X^2$ and are orthogonal to $L_2(O)$. If $H$ denotes the mean curvature of $X^2$ and $d\sigma$ denotes the element of area of $X^2$, it is known that (34) is equivalent to the *total absolute mean curvature*

$$K_{1,1}^{(1)}(X^2) = \frac{1}{2} \int_{X^2} |H|\, d\sigma. \tag{35}$$

(*b*) *Case* $r = 2$, $N = 1$. Surface $X^2 \subset E_3$. The Grassman manifold $G_{1,2}$ is the unit sphere $\Sigma_2$ and equation (25) can be written

$$K_{2,1}^{(1)}(X^2) = \frac{1}{4\pi} \int_{\Sigma_2} \nu_2\, dL_1(O), \tag{36}$$

where $\nu_2$ is the number of planes in $E_3$ which are tangent to $X^2$ and are orthogonal to the line $L_1(O)$. If $K$ denotes the Gaussian curvature of $X^2$, since $dL_1(O)$ is the element of area on $\Sigma_2$, it is easy to see that equation (36) is equivalent, up to a constant factor, to the *total absolute Gaussian curvature* of $X^2$, that is,

$$(37) \qquad K_{2,1}^{(1)}(X^2) = \frac{1}{2\pi}\int_{X^2}|K|\,d\sigma.$$

(*c*) *Case* $r = 1$, $N = 2$. Surfaces $X^2 \subset E_4$. In this case, writing $\Sigma_3$ = unit 3-dimensional sphere, instead of $G_{3,1}$, we have

$$(38) \qquad K_{1,3}^{(1)}(X^2) = \frac{1}{2\pi^2}\int_{\Sigma_3}\nu_1\,dL_3(O),$$

where $\nu_1$ is the number of tangent lines to $X^2$ which are orthogonal to the hyperplane $L_3(O)$. The properties of this total absolute curvature seem not to be known. A geometrical interpretation was given in [10].

(*d*) *Case* $r = N+1$. Surfaces $X^2 \subset E_{N+2}$. According to (25) we have the following curvature

$$(39) \qquad K_{N+1,N}^{(1)}(X^2) = \frac{1}{O_{N+1}}\int_{\Sigma_N}\nu_{N+1}\,dL_1(O),$$

where $\nu_{N+1}$ is the number of hyperplanes of $E_{N+2}$ which are tangent to $X^2$ and are orthogonal to the line $L_1(O)$ and $\Sigma_N$ denotes the $N$-dimensional unit sphere. Up to a constant factor this curvature coincides with the *curvature of Chern–Lashof* [3]. Since obviously $\nu_{N+1} \geq 2$ we have the inequality $K_{N+1,N}^{(1)} \geq 2$, with the equality sign only if $X^2$ is a convex surface contained in a linear subspace $L_3$ of $E_4$.

For $N = 2$, $X^2$ is a surface imbedded in $E_4$ and the curvature (39) is a kind of dual of the curvature (38) (see [10]).

(*ii*) *Total absolute curvatures of order* $q = 2$. We have $n = 2$, $\rho = 5$ and the inequalities (26) and (27) say that the only possible cases are: (*a*) $r = 1$, $N = 4$; (*b*) $r = 2$, $N = 4$; (*c*) $r = 1$, $N = 5$.

(*a*) *Case* $r = 1$, $N = 4$. Surface $X^2$ in $E_6$. The Grassman manifold $G_{5,1}$ is the unit sphere $\Sigma_5$ and equation (25) can be written

$$(40) \qquad K_{1,4}^{(2)}(X^2) = \frac{1}{O_5}\int_{\Sigma_5}\lambda\,dL_5(O),$$

where $\lambda$ is the length of the curve in $L_5(O)$ generated by the intersections of $L_5(O)$ with the lines of $E_6$ which are orthogonal to $L_5(O)$ and belong to some of the second tangent fibres of $X^2$.

(*b*) *Case* $r = 2$, $N = 4$. Surface $X^2$ in $E_6$. We have

$$(41) \qquad K_{2,4}^{(2)}(X^2) = \frac{O_1}{O_4 O_5}\int_{G_{4,2}}\nu_2\,dL_4(O),$$

where $\nu_2$ is the number of 2-spaces of $E_6$ which are orthogonal to $L_4(O)$ and are contained in some second tangent fibre of $X^2$.

(c) *Case* $r = 1$, $N = 5$. Surfaces $X^2$ in $E_7$.

We have

$$K_{1,5}^{(2)}(X^2) = \frac{1}{O_6} \int_{\Sigma_6} \nu_1 \, dL_6(O), \tag{42}$$

where $\nu_1$ is the number of lines of $E_7$ which are contained in some second tangent fibre of $X^2$ and are orthogonal to $L_6(O)$.

The expression of these absolute total curvatures of order 2 by means of differential invariants of $X^2$ is not known.

## REFERENCES


1. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, *Erg. der Mathematik*, Berlin (1934).
2. S. S. Chern, "On the kinematic formula in integral geometry, *J. Math. Mechanics*, 16 (1966), 101–118.
3. S. S. Chern and R. K. Lashof, "On the total curvature of immersed manifolds", *Am. J. Math.* 79 (1957), 306–318.
4. L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton (1949).
5. H. Federer, "Curvature measures", *Trans. Am. Math. Soc.* 93 (1959), 418–491.
6. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin (1957).
7. N. Kuiper, "Der Satz von Gauss-Bonnet für Abbildungen in $E^N$ und damit verwandte Probleme", *Jahr. Deutsch. Math. Ver.* 69 (1967), 77–88.
8. L. A. Santaló, "Geometría integral en espacios de curvatura constante", *Publicaciones Com. Energia Atómica, Serie Matem. Buenos Aires* (1952).
9. ——, "Sur la mesure des espaces linéaires qui coupent un corps convexe et problèmes que s'y rattachent", *Colloque sur les Questions de Réalité en Géométrie, Liege*, (1955), 177–190.
10. ——, "Curvaturas absolutas totales de variedades contenidas en un espacio euclidiano", *Coloquio de Geometria Diferencial, Santiago de Compostela (España)* (1967), pp. 29–38.