

ON THE SUPERPOSITION OF RANDOM MOSAICS

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ABSTRACT

We compute the mean values of the area $a_{12\dots m}$, the perimeter $h_{12\dots m}$, the number of arcs $w_{12\dots m}$ and the number of vertices $v_{12\dots m}$ of a typical polygon of the superposition of m independent random mosaics. Some particular cases are considered. For definitions and basic formulas see Cowan (1980) and Santaló (1984).

SUPERPOSITION OF RANDOM MOSAICS

Cowan (1980) defines as characteristics of a random mosaic M , the mean values of the area a , the perimeter h , the number of arcs w and the number of vertices v of a "typical polygon" of M (suitably defined).

In Santaló (1984) we computed the characteristics of the random mosaic obtained by homogeneous random superposition of two independent random mosaics M_i of characteristics a_i, h_i, w_i, v_i ($i = 1, 2$). The result was

$$\begin{aligned} a_{12} &= \frac{2\pi a_1 a_2}{2\pi(a_1 + a_2) + h_1 h_2}, & h_{12} &= \frac{2\pi(a_1 h_2 + a_2 h_1)}{2\pi(a_1 + a_2) + h_1 h_2} \\ w_{12} &= \frac{2\pi(w_1 a_2 + w_2 a_1) + 4h_1 h_2}{2\pi(a_1 + a_2) + h_1 h_2} \\ v_{12} &= \frac{2\pi(v_1 a_2 + v_2 a_1) + 4h_1 h_2}{2\pi(a_1 + a_2) + h_1 h_2} \end{aligned} \tag{1}$$

If we superpose m independent random mosaics M_i ($i = 1, 2, \dots, m$) of characteristics a_i, h_i, w_i, v_i (always assuming that the superposition is random homogeneous), we get the following result:

Theorem 1. *The characteristics of the random mosaic obtained by superposition of*

m independent random mosaics M_i , are the following:

$$\begin{aligned} a_{1\dots m} &= \Delta^{-1} 2\pi a_1 \dots a_m \\ h_{1\dots m} &= \Delta^{-1} 2\pi \{h_1 \mid a_2 \dots a_m\} \\ w_{1\dots m} &= \Delta^{-1} (2\pi \{w_1 \mid a_2 \dots a_m\} + 4\{h_1 h_2 \mid a_3 \dots a_m\}) \\ v_{1\dots m} &= \Delta^{-1} (2\pi \{v_1 \mid a_2 \dots a_m\} + 4\{h_1 h_2 \mid a_3 \dots a_m\}) \end{aligned} \quad (2)$$

where

$$\Delta = 2\pi \{a_1 \dots a_{m-1}\} + \{h_1 h_2 \mid a_3 \dots a_m\}$$

and $\{ \}$ indicates "symmetric functions", i.e.

$$\begin{aligned} \{h_1 \mid a_2 \dots a_m\} &= h_1 a_2 \dots a_m + a_1 h_2 \dots a_m + \dots + a_1 \dots a_{m-1} h_m \\ \{h_1 h_2 \mid a_3 \dots a_m\} &= h_1 h_2 a_3 \dots a_m + h_1 a_2 h_3 \dots a_m + \dots + a_1 \dots h_{m-1} h_m \\ \{a_1 \dots a_{m-1}\} &= a_1 a_2 \dots a_{m-1} + a_1 \dots a_{m-2} a_m + \dots + a_2 a_3 \dots a_m. \end{aligned}$$

Proof. By induction. For $m = 2$ the formulas (2) hold, since they coincide with (1). Assuming that they hold for m mosaics M_i applying (1) to the pair of mosaics $M_1 \cup M_2 \cup \dots \cup M_m$ and M_{m+1} , a straightforward computation verifies that (2) holds for $m + 1$ mosaics.

CASE OF MOSAICS WITH THE SAME CHARACTERISTICS

If the random mosaics have the same characteristics a, h, w, v the formulas (2) take the form

$$\begin{aligned} a_m &= 4(m\Delta)^{-1} \pi a^2, \quad h_m = 4\Delta^{-1} \pi a h \\ w_m &= 4\Delta^{-1} (\pi a w + (m-1)h^2) \\ v_m &= 4\Delta^{-1} (\pi a v + (m-1)h^2), \end{aligned} \quad (3)$$

where

$$\Delta = 4\pi a + (m-1)h^2.$$

Consequences. 1. If $v = 4$, we have $v_m = 4$ for any m .

2. For $m \rightarrow \infty$ we always have $v_m \rightarrow 4$.

Examples. 1. For Poisson random mosaics (Miles, 1970; Santaló, 1976, p.57) we have $a = 4/\pi\lambda^2$, $h = 4/\lambda$, $w = v = 4$ and (3) gives a_m, h_m, w_m, v_m . For instance, we have $w_m = v_m = 4$ for any m .

2. For random mosaics of Voronoi type of the same characteristics, we have (Miles, 1970; Santaló, 1976, p.57), $a = 1/\lambda$, $h = 4/\lambda^{1/2}$, $w = v = 6$ and we have

$$w_m = v_m = 6 - \frac{8(m-1)}{\pi + 4(m-1)}$$

which is a decreasing function of m , from 6 to 4.

3. For random mosaics of Delaunay type, see Miles (1970), we have $a = 1/2\lambda$, $h = 32/9\pi\sqrt{\lambda}$, $w = v = 3$ and we get

$$(2) \quad w_m = v_m = 3 + \frac{32^2(m-1)}{32^2(m-1) + 162\pi^3}$$

4. Consider the regular mosaics of equilateral triangles ($w = v = 3$), squares ($w = v = 4$), regular hexagons ($w = v = 6$), or any affine transforms of them. By uniform random superposition of m such mosaics we get, respectively (according to (3)),

$$\begin{aligned} w_m = v_m \text{ (triangles)} &= 4 - \frac{4\pi a}{4\pi a + (m-1)h^2} \\ w_m = v_m \text{ (parallelograms)} &= 4 \\ w_m = v_m \text{ (hexagons)} &= 4 + \frac{8\pi a}{4\pi a + (m-1)h^2} \end{aligned}$$

i.e. the mean number of vertices (equal to the mean number of sides) of a typical polygon is less than 4 for the superposition of triangular mosaics, equal to 4 for the superposition of mosaics of parallelograms and greater than 4 for the superposition of mosaics of hexagons. This gives a criterion for recognising if a given random mosaic is the result of superposition of mosaics of triangles, parallelograms or hexagons. Of course, the condition of w or v being less, equal or greater than 4 is only a necessary condition, not sufficient.

MOSAICS OF RECTANGLES

Formulas (1) apply to the mosaics obtained by random superposition of m non random mosaics (tessellations, i.e. arrangements of congruent polygons fitting together so as to cover the whole plane without overlapping). Then a_i , h_i , w_i , v_i are the area, the perimeter, the number of sides and the number of vertices of a polygon of the mosaic. The mosaics can be assumed moving in the plane without deformation with the kinematic density of integral geometry (Santaló, 1976).

Consider, for instance, the case of m mosaics M_i of congruent rectangles of sides δ_i , λ_i ($i = 1, 2, \dots, m$) (formed by lines parallel to the x -axis at distance δ_i apart and the lattice of orthogonal parallel lines at distance λ_i apart). Then we have

$$a_i = \delta_i \lambda_i; \quad h_i = 2(\delta_i + \lambda_i), \quad w_i = v_i = 4$$

and for the mosaic obtained by random superposition of them we get

$$\begin{aligned} a_{12\dots m} \text{ (rectangles)} &= \pi \Delta^{-1} \delta_1 \dots \delta_m \lambda_1 \dots \lambda_m \\ h_{12\dots m} \text{ (rectangles)} &= 2\pi \Delta^{-1} \{(\delta_1 + \lambda_1)\delta_2 \lambda_2 \dots \delta_m \lambda_m\} \\ w_{12\dots m} = v_{12\dots m} \text{ (rectangles)} &= 4, \end{aligned} \tag{4}$$

where

$$\Delta (\text{rectangles}) = \pi \{ \delta_1 \lambda_1 \dots \delta_{m-1} \lambda_{m-1} \} 2 \{ (\delta_1 + \lambda_1)(\delta_2 + \lambda_2) \delta_3 \lambda_3 \dots \delta_m \lambda_m \} .$$

For congruent mosaics of rectangles of sides $\delta_i = \delta$, $\lambda_i = \lambda$ we have

$$a_{1\dots m} (\text{congruent rectangles}) = \frac{\pi \delta^2 \lambda^2}{\pi m \delta \lambda + m(m-1)(\delta + \lambda)^2}$$

$$h_{1\dots m} (\text{congruent rectangles}) = \frac{2\pi(\delta + \lambda)\delta\lambda}{\pi\delta\lambda + (m-1)(\delta + \lambda)^2}$$

$$w_{1\dots m} = v_{1\dots m} (\text{congruent rectangles}) = 4 .$$

For mosaics of squares of side δ we have $\lambda = \delta$ and thus

$$a_{1\dots m} (\text{squares}) = \frac{\pi \delta^2}{\pi m + 4m(m-1)}$$

$$h_{1\dots m} (\text{squares}) = \frac{4\pi\delta}{\pi + 4(m-1)}$$

$$w_{1\dots m} = v_{1\dots m} (\text{squares}) = 4 .$$

If $\lambda_1, \lambda_2, \dots, \lambda_m \rightarrow \infty$ the mosaics of rectangles tend to lattices of parallel lines at distances $\delta_1, \delta_2, \dots, \delta_m$ apart. Then, from (4) we deduce the following.

Theorem 2. *If m lattices of parallel lines at distances $\delta_1, \delta_2, \dots, \delta_m$ apart are superposed independently at random, the resulting random mosaic has the following characteristics*

$$a_{1\dots m} (\text{parallel lines}) = \pi \Delta^{-1} \delta_1 \dots \delta_m$$

$$h_{1\dots m} (\text{parallel lines}) = 2\pi \Delta^{-1} \{ \delta_2 \dots \delta_m \}$$

$$w_{1\dots m} = v_{1\dots m} (\text{parallel lines}) = 4 ,$$

where

$$\Delta = 2 \{ \delta_1 \delta_2 \dots \delta_m \} .$$

If the parallel lines are the same distance apart for all lattices, we have $\delta_1 = \delta_2 = \dots = \delta_m = \delta$ and so

$$a_{1\dots m} (\text{equidistant parallel lines}) = \frac{\pi \delta^2}{m(m-1)}$$

$$h_{1\dots m} (\text{equidistant parallel lines}) = \frac{2\pi\delta}{m-1}$$

$$w_{1\dots m} = v_{1\dots m} (\text{equidistant parallel lines}) = 4 .$$

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