Sonderabdruck aus
ARCHIV DER MATHEMATIK
Vol. XVIII, 1967 BIRKHÄUSER VERLAG, BASEL UND STUTTGART Fasc. 5

Horocycles and Convex Sets in Hyperbolic Plane

By

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1. Introduction. The study of the integral geometry in hyperbolic plane was carried out in [2]; see also [3]. However, the basic elements there considered were only points, lines and sets of congruent figures. The horocycles were not considered, in spite of the important role they play in hyperbolic plane geometry. The purpose of the present paper is to fill this gap, by defining a density for sets of horocycles and obtaining the integral formulae (3.2), (4.1), (5.8), (5.11) which generalize to horocycles certain known formulae for convex sets and straight lines. As usual in integral geometry, we call "density" any differential form whose integral gives an invariant measure under the group of hyperbolic motions.

2. Density for horocycles. In terms of the polar coordinates r, φ the line element of the hyperbolic plane has the form

$$ds^2 = dr^2 + \sinh^2 r \, d\varphi^2 \tag{2.1}$$

and the area element is

$$df = \sinh r \, dr \wedge d\varphi . \tag{2.2}$$

Let C be a circle of radius R and center $C_1(r, \varphi)$. Denoting by ϱ the distance from the origin of coordinates O to C, the area element corresponding to C_1 will be $dC_1 = \sinh(\varrho + R)\, d\varrho \wedge d\varphi$ if O is exterior to C and $dC_1 = \sinh(R - \varrho)\, d\varrho \wedge d\varphi$ if O is interior to C.

By fixed R, the product $f(R)\, dC_1$ is invariant, for any $f(R)$, by the group of hyperbolic motions and therefore it can be taken as a density for sets of circles of radius R. As $R \to \infty$ the circle C tends to the horocycle $H(\varrho, \varphi)$ and in order that $f(R)\, dC_1$ approaches to a limit ($\neq 0, \infty$) we must take $f(R)$ such that $f(R)\, e^R \to a$, a being a constant which for simplicity we assume equal to 2. Then, if we denote by dH_+ the density for horocycles which turn the convexity towards O and by dH_- the density for horocycles which turn the convexity towards the opposite sense, we have

$$dH_+ = e^{\varrho}\, d\varrho \wedge d\varphi , \quad dH_- = e^{-\varrho}\, d\varrho \wedge d\varphi . \tag{2.3}$$

This density, that we will denote indistinctly by dH, is uniquely determined, i.e. it is unique, up to a constant factor, which is invariant under the group of hyperbolic motions, as follows from the way we have obtained it.

3. Horocycles which intersect a curve. Let Q be a rectifiable curve of length L. Then, the so-called Poincaré's formula of the integral geometry applied to Q and to the circle C of radius R writes [2] (having into account that the length of C is $2\pi \sinh R$)

$$\int n \, dC_1 = 4 L \sinh R \tag{3.1}$$

where n is the number of intersection points of Q and C, the integral extended over the whole hyperbolic plane, n being zero if Q and C do not intersect. Multiplying (3.1) by $2e^{-R}$ and letting R tend to infinity, we get

$$\int n \, dH = 4 L \tag{3.2}$$

where n is now the number of intersection points of Q and the horocycle H, the integral being extended over all horocycles of the plane.

Notice that (3.2) does not change if horocycles are substituted by oriented lines [2].

4. Horocycles which intersect a *h*-convex set. A set of points K in the hyperbolic plane is said to be convex if for each pair of points A, B belonging to K, the entire segment of straight line AB also belongs to K.

A set of points K is said to be h-convex or convex with respect to horocycles, if for each pair of points A, B belonging to K, the entire segments of the two horocycles AB also belong to K.

Two points A, B of the hyperbolic plane determine two horocycles H, H' which contain these points. If K is h-convex the whole lune bounded by H and H' belongs to K and therefore the line segment AB also belongs to K, i.e. any h-convex set is convex. The converse is not true, as is immediately shown by any convex set containing a line segment in its boundary. Since the curvature of the horocycles is equal to 1, it is clear that any convex set bounded by a smooth curve of curvature greater or equal than 1 at every point is h-convex.

Since the set of support horocycles (likewise as the set of support lines) of a h-convex set is a set of measure zero, by applying (3.2) to the boundary of K we will have $n = 2$ up to a set of zero measure, and therefore we get

$$\int_{H \cap K \neq 0} dH = 2L. \tag{4.1}$$

Thus we have: *the measure of the set of horocycles which intersect a h-convex set K is equal to $2L$, where L is the length of the boundary of K.*

5. Density for pairs of points and integral formula for chords. Let K be a convex set of the hyperbolic plane and let dG represent the density for lines. If σ denotes the length of the chord that G determines on K, i.e. the length of the intersection $G \cap K$, the following formulae are known [2],

$$\int_{G \cap K \neq 0} \sigma \, dG = \pi F, \quad \int_{G \cap K \neq 0} \sinh \sigma \, dG = \pi F + \tfrac{1}{2} F^2 \tag{5.1}$$

where F is the area of K.

In the euclidean plane the first formula (5.1) holds without change, while the second gives rise to the so-called Crofton's formula for chords, which writes [2]

$$\int\limits_{G \cap K \neq 0} \sigma^3 \, dG = 3 F^2 . \tag{5.2}$$

We wish now to see what happens in the formulae (5.1) when lines are substituted by horocycles. In order to do that we need a formula which gives the product $dP_1 \wedge dP_2$ of the densities of two points P_1, P_2 (area elements at P_1, P_2) in terms of the density dH of a horocycle H determined by those points and the differentials dt_1, dt_2 of the abscissae t_1, t_2 of P_1, P_2 on H.

Let us consider first a circle C (of center $C_1(\varrho, \varphi)$ and radius R) which passes through $P_1(r_1, \varphi + \psi_1)$, $P_2(r_2, \varphi + \psi_2)$ (Fig. 1).

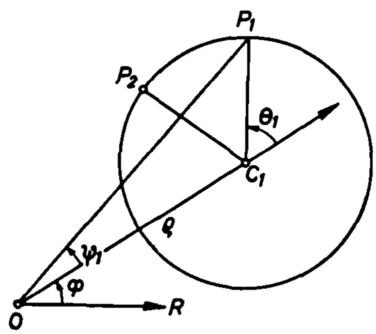

Fig. 1

If θ_1 denotes the angle which forms the radius $C_1 P_1$ with the line OC_1 and ψ_1 the angle between OC_1 and OP_1, by well known formulae of hyperbolic geometry [1, p. 237] we have

$$\cosh r_1 = \cosh R \cosh \varrho + \sinh R \sinh \varrho \cos \theta_1 ,$$
$$\sin \psi_1 \sinh r_1 = \sinh R \sin \theta_1 .$$

Differentiating we get

$$\begin{aligned} \sinh r_1 \, dr_1 &= (\cosh R \sinh \varrho + \sinh R \cosh \varrho \cos \theta_1) \, d\varrho - \sinh R \sinh \varrho \sin \theta_1 \, d\theta_1 , \\ \cos \psi_1 \sinh r_1 \, d\psi_1 &+ \sin \psi_1 \cosh r_1 \, dr_1 = \sinh R \cos \theta_1 \, d\theta_1 . \end{aligned} \tag{5.3}$$

The last equation can be written

$$\begin{aligned} \cos \psi_1 \sinh r_1 \, d(\psi_1 + \varphi) + \sin \psi_1 \cosh r_1 \, dr_1 &= \\ = \sinh R \cos \theta_1 \, d\theta_1 + \cos \psi_1 \sinh r_1 \, d\varphi . \end{aligned} \tag{5.4}$$

Exterior multiplication of (5.3) and (5.4), putting

$$dP_1 = \sinh r_1 \, dr_1 \wedge d(\psi_1 + \varphi) = \text{area element at } P_1 ,$$

gives

$$\begin{aligned} &\sinh r_1 \cos \psi_1 \, dP_1 = \\ &= (\cosh R \sinh \varrho + \sinh R \cosh \varrho \cos \theta_1)(\sinh R \cos \theta_1 \, d\varrho \wedge d\theta_1 + \cos \psi_1 \sinh r_1 \, d\varrho \wedge d\varphi) \\ &\quad - \sinh R \sinh \varrho \sin \theta_1 \cos \psi_1 \sinh r_1 \, d\theta_1 \wedge d\varphi . \end{aligned}$$

An analogous formula holds for P_2 and by exterior multiplication we get

$$\begin{aligned}(5.5)\qquad &\sinh r_1 \sinh r_2 \cos\psi_1 \cos\psi_2\, dP_1 \wedge dP_2 = \\ &= [\sin\theta_1 \cos\theta_2 \cos\psi_1 \sinh r_1 (\cosh R \sinh\varrho + \sinh R \cosh\varrho \cos\theta_2) - \\ &\quad - \sin\theta_2 \cos\theta_1 \cos\psi_2 \sinh r_2 (\cosh R \sinh\varrho + \sinh R \cosh\varrho \cos\theta_1)] \times \\ &\quad \times \sinh^2 R\, dC_1 \wedge d\theta_1 \wedge d\theta_2\,,\end{aligned}$$

where $dC_1 = \sinh\varrho\, d\varrho \wedge d\varphi$ is the area element corresponding to C_1.

By a known formula of hyperbolic trigonometry we have, for $i = 1, 2$,

$$\sinh r_i \cos\psi_i = \cosh R \sinh\varrho + \sinh R \cosh\varrho \cos\theta_i$$

and therefore, (5.5) gives

$$(5.6)\qquad dP_1 \wedge dP_2 = |\sin(\theta_2 - \theta_1)| \sinh^2 R\, dC_1 \wedge d\theta_1 \wedge d\theta_2$$

where we have put $|\sin(\theta_2 - \theta_1)|$ since we always consider the densities in absolute value.

Instead of the angles θ_1, θ_2 we can introduce the abscissae t_1, t_2 of P_1, P_2 on the circumference of C related by the equations

$$dt_1 = \sinh R\, d\theta_1\,, \quad dt_2 = \sinh R\, d\theta_2\,, \quad t_2 - t_1 = (\theta_2 - \theta_1) \sinh R\,.$$

Substituting in (5.6) and letting R tend to infinity, since $2e^{-R} dC_1 \to dH$ = density for horocycles, we get

$$(5.7)\qquad dP_1 \wedge dP_2 = |t_2 - t_1|\, dH \wedge dt_1 \wedge dt_2\,.$$

This formula is the same as the formula for pairs of points in euclidean plane [3] and therefore, integrating over all pairs of points inside a h-convex set K, we obtain

$$(5.8)\qquad \int\limits_{H \cap K \neq \emptyset} \sigma^3\, dH = 6F^2$$

where σ is the length of the chord $H \cap K$, assumed K h-convex. This formula (5.8) differs from the formula (5.2) for chords in euclidean plane by a factor 2, due to the fact that two points determine two horocycles.

A kind of dual formula is obtained from (5.6) if we consider P_1, P_2 as centers of two circles of radius R and let $R \to \infty$. Then we get two horocycles H_1, H_2 which intersect at the point C_1 under the angle $\theta_2 - \theta_1$ and the differential formula

$$(5.9)\qquad dH_1 \wedge dH_2 = |\sin(\theta_2 - \theta_1)|\, dC_1 \wedge d\theta_1 \wedge d\theta_2$$

holds.

This formula (5.9) does not change if horocycles are substituted by lines [2] and has the same form as in euclidean plane [3].

In order to generalize the first formula (5.1) to horocycles, let us integrate both sides of (5.9) over all the pairs of horocycles which intersect each other in the interior of a domain K (not necessarily convex) of area F. Since there are two horocycles tangent to a given direction at a point, the right side gives

$$(5.10)\qquad 4 \int\limits_{C_1 \in K} dC_1 \int\limits_0^\pi \int\limits_0^\pi |\sin(\theta_2 - \theta_1)|\, d\theta_1 \wedge d\theta_2 = 8\pi F\,.$$

In this computation, if the horocycles H_1, H_2 have two intersection points in K, the pair H_1, H_2 has been counted two times. Therefore if we call σ_1 the length of the arc of H_1 which belongs to K, in accordance with (3.2) the integral of dH_2 over all H_2 which cut H_1 in a point of K, is $4\sigma_1$. Thus the integral of the left of (5.9) is $4\int \sigma_1 dH_1$. Equating to (5.10) and writing σ and H in place of σ_1 and H_1, we get

$$\int_{H \cap K \neq \emptyset} \sigma\, dH = 2\pi F \tag{5.11}$$

which is the generalization we wish to obtain. Note that in (5.11) K is not necessarily convex.

Bibliography

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Eingegangen am 18. 11. 1966

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