RANDOM PROCESSES OF LINEAR SEGMENTS AND GRAPHS

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SUMMARY

By a graph G we understand a finite set of points (vertices) together with the line segments which unites some pairs of distinct points of the set. Sets of congruent graphs are considered. The position of a graph on the plane is defined by the position of one of its vertices P and a rotation ϕ about P. Assuming P Poisson distributed on the plane and ϕ uniformly distributed over $0 \leq \phi < 2\pi$, we extend to graph processes some known properties of line segment processes (Coleman [1], [2] ; Parker and Cowan [3]). We find the probability that the distance from a point chosen at random independently of the process of graphs to the nearest vertice of a graph or to the nearest graph exceeds u . Some of the results are also extended from the euclidean plane to surfaces (sets of geodesic segments and sets of geodesic graphs), for instance to the sphere and to the hyperbolic plane.

1. INTRODUCTION

An oriented line segment S of length s , may be defined on the plane by its origin P(x,y) and the angle ϕ that it makes with a fixed direction, for instance with the x-axis. If the length s is random variable with probability density function f(s), so that

$$(1.1) \qquad \int_0^\infty f(s)\, ds = 1 , \qquad \int_0^\infty s\, f(s)\, ds = E(s)$$

the density for sets of uniformly distributed oriented line segments is defined by any one of the following equivalent differential forms [5]

$$(1.2) \qquad dS = f(s)\, ds \wedge dP \wedge d\phi = f(s)\, ds \wedge dG \wedge dt$$

where dP means the area element at P , dG is the density for

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oriented straight lines (corresponding to the line that contains S) and t denotes the abscissa of P on G. The densities are always considered in absolute value, so that the order of the differentials in the forms above is inmaterial. All the lengths and orientations of the segments are mutually independent.

With these assumptions, Coleman [1], [2] and Parker and Cowan [3], have considered random processes of line segments on the plane of intensity λ (mean number of points P per unit area). Though Parker and Cowan have considered more general processes, we shall assume, following Coleman, that the process of points P is a homogeneous Poisson process of intensity λ. We state some of their results:

i) The probability that the distance from a point chosen at random independently of the process of line segments to the nearest origin or end of a line segment exceeds u ($0 \leq u < \infty$) is $\exp(-\lambda H)$, where

$$(1.3)\quad H = 2u^2\left(\pi - \int_0^{2u} (\arccos(s/2u) - (s/2u)(1-s^2/4u^2)^{1/2}) f(s)\, ds\right).$$

ii) The mean value of the number ν of origin or end points of the line segments that are contained in a convex set K of area F and perimeter L which is chosen at random in the plane (in Fig.1 is $\nu=15$), is

$$(1.4)\qquad E(\nu) = 2\lambda F .$$

iii) The mean value of the number m of line segments which have common point with K (in Fig.1 is $m=10$), is

$$(1.5)\qquad E(m) = \lambda(F + \pi^{-1}E(s)L)$$

iv) The mean value of the number m^* of intersection points of line segments with a rectifiable curve of length L chosen at random on the plane (uniformly distributed and independent of the process), is

$$(1.6)\qquad E(m^*) = 2\lambda\pi^{-1}E(s)L.$$

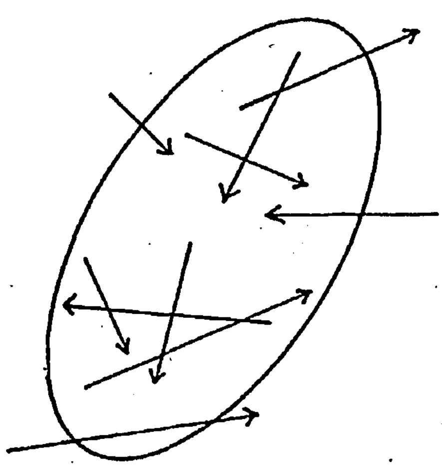

Fig.1

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v) The mean value of the number N of segment-segment crossings within K (in Fig.1 is $N=6$), is

(1.7) $$E(N) = \pi^{-1}\lambda^2 (E(s))^2 F .$$

vi) The mean value of the total length within K of segments which intersect K, is

(1.8) $$E(\Sigma\alpha_i) = \lambda E(s) F.$$

More generally we can prove that, assuming s greater than the diameter of K,

(1.9) $$E(\Sigma\alpha_i^n) = \pi^{-1}\lambda\left(E(s)\, I_n - \frac{n-1}{n+1} I_{n+1}\right)$$

where I_n are the invariants of the convex set K defined by

(1.10) $$I_n = \int \sigma^n \, dG$$

where σ is the length of the chord $G \cap K$ and the integral is extended over all the lines G of the plane.

vii) The probability that the distance from a point Q chosen at random independently of the process to the nearest line segment exceeds u $(0 \leq u < \infty)$ is $\exp(-\lambda H_1(u))$, where

(1.11) $$H_1(u) = \pi u^2 + 2uE(s) .$$

These results can be extended in three different directions: 1. Extension to random figures other than line segments; 2. Extension from the euclidean plane to other surfaces, for instance the sphere or the hyperbolic plane, 3. Extension from the euclidean plane E_2 to the euclidean space E_n.

In this paper we will be concerned with the cases 1 and 2. The extension to E_n presents a great deal of possibilities and will be considered elsewhere.

2. FIRST EXTENSION: SETS OF RANDOM GRAPHS.

2.1 Definitions and some mean values. A graph T consists of a finite non empty set of points (vertices) together with a prescribed set of line segments (arcs) which join some pairs of vertices. Let v be the number of vertices and h the number of arcs. We consi-

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der sets of similar graphs, i.e. graphs which can be mapped one to another by a similitude, and we will denote by s the scale factor of the similitude. The lengths of the arcs are denoted by $sa_1, sa_2, \ldots, sa_h$, so that the total length of the graph is $A=s(a_1+a_2+\ldots+a_h)$.

A graph T is defined in the plane, up to an isometry, by one of its vertices, say $P(x,y)$, the scale factor s and a rotation about P through the angle ϕ. We will consider sets of independent random graphs such that P is uniformly distributed on the plane with intensity λ (number of points P per unit area), ϕ is uniformly distributed over the range $0 \leq \phi < 2\pi$, and the scale factor s has a probability density function $f(s)$ which satisfies conditions (1.1). This means that the so called density for sets of graphs is the differential form (1.2), which we now write

(2.1) $$dT = f(s)\, ds \wedge dP \wedge d\phi \ .$$

The mean length of the graphs T is $E(A) = (a_1+a_2+\ldots+a_h)E(s)$.

Applying that the expectation of the sum is the sum of expectations (provided they exist), some of the mean values of the Introduction generalize immediately to random graphs. For instance:

a) Denoting by ν the number of vertices which are contined in a convex set K of area F and perimeter L placed at random on the plane, we have

(2.2) $$E(\nu) = \lambda v F \ ;$$

b) The mean value of the number m^* of intersection points of a rectifiable curve of length L placed at random on the plane, with the arcs of the graphs, is

(2.3) $$E(m^*) = 2\pi^{-1}\lambda\, E(A)\, L$$

which generalizes (1.6).

c) The mean value of the total length within K of arcs that intersect with a convex set K placed at random on the plane, is

(2.4) $$E(\Sigma A_i) = \lambda\, F\, E(A) \ ,$$

which generalizes (1.8).

d) The mean value of the number N of arcs-arcs of graphs crossing within a convex set K, is

$$E(N) = \pi^{-1} \lambda^2 (E(A))^2 F . \tag{2.5}$$

2.2 <u>The distribution of the distances from a given point to the nearest vertex of a random process of graphs</u>. For simplicity, we shall consider now a process of isometric graphs. We will represent the lengths of the arcs by a_i, so that the total length is $A=a_1+a_2+\ldots+a_h$. Round each vertex of the graph T we construct

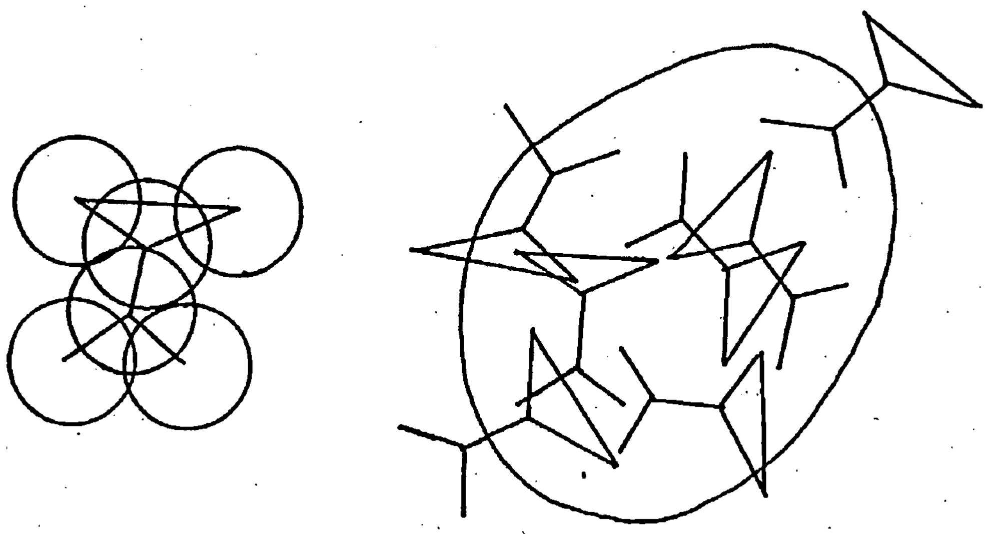

Fig.2

a disc of radius u (Fig.2). Let D_i $(i=1,2,\ldots,v)$ be the set of those points which are covered exactly by i discs and let F_i denote its area. Assuming that $F_1, F_2,\ldots,F_m$ are $\neq 0$ and that $F_{m+1}=\ldots=F_v = 0$, we put

$$F=F_1+F_2+\ldots+F_m = \pi u^2 v - F_2 - 2F_3 - \ldots -(m-1)F_m . \tag{2.6}$$

The functions $F_i(u)$ are characteristics of the graph. It should be interesting to know until which extent they determine T.

Consider n random graphs (with the discs included) which have the vertex P inside a large disc of radius R. The probability that a chosen point Q of the disc (sufficiently far from the boundary) be covered exactly by r_1 sets D_1, r_2 sets $D_2,\ldots,$ r_m sets

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D_m $(n \geq \Sigma r_i,\ i=1,2,\dots,m)$ is (multinomial distribution)

$$(2.7)\quad p^{(n)}_{r_1\dots r_m} = \frac{n!}{r_1!\dots r_m!(n-\Sigma r_i)!}\left(\frac{F_1}{F_0}\right)^{r_1}\dots\left(\frac{F_m}{F_0}\right)^{r_m}\left(1-\frac{F}{F_0}\right)^{n-\Sigma r_i}$$

where $F_0 = \pi R^2$. If n and $R \to \infty$ in such a way that $n/F_0 \to \lambda$ (number of graphs per unit area), the probability tends to the limit

$$(2.8)\qquad p_{r_1\dots r_m} = \frac{(\lambda F_1)^{r_1}}{r_1!}\ \dots\ \frac{(\lambda F_m)^{r_m}}{r_m!}\exp(-\lambda F)$$

which is a multiple Poisson distribution. The obtained process is called a Poisson graph process of intensity λ . Thus we have proved that

Consider a Poisson process of congruent graphs T of intensity λ . The probability that the distance from a point Q chosen at random independently of the process to i vertices of r_i graphs $(i=1,2,\dots,m)$ does not exceed u is given by (2.8).

In particular, the probability that the distance from Q to the nearest vertex exceeds u , is

$$(2.9)\qquad p_{0\dots 0} = \exp(-\lambda F).$$

The function $F(u)$ is in general difficult to calculate. By a direct computation, we can find:

a) If T is a line segment of length a , we have

$$(2.10)\qquad F = \pi u^2 \qquad \text{if}\quad a \geq 2u$$

$$F = 2u^2\{\pi - \arccos(a/2u) + (a/2u)(1 - a^2/4u^2)^{1/2}\}\quad \text{if}\quad a \leq 2u$$

b) If T is a rectangle of sides a, b such that $b \geq a$, we have (Fig.3,a,b,c,d)

$$(2.11)\qquad F = 4\pi u^2 \qquad \text{if}\quad 2u \leq a\ ;$$

$$F = 4u^2\{\pi - \arccos(a/2u) + (a/2u)(1-a^2/4u^2)^{1/2}\}\quad \text{if}\quad a \leq 2u \leq b\ ;$$

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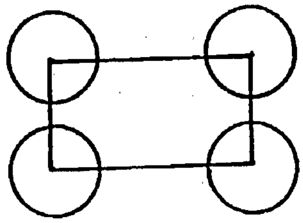

Fig.3, a)

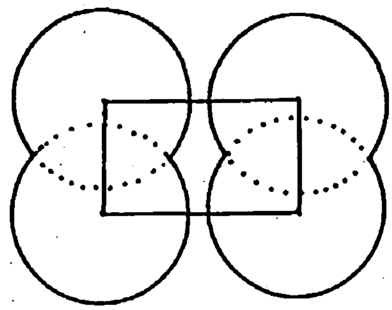

Fig.3, b)

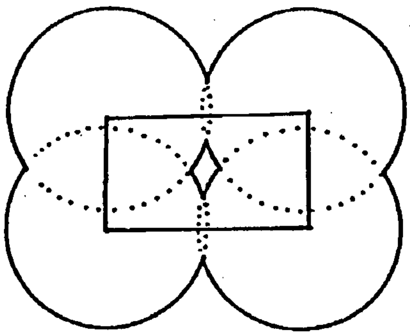

Fig.3, c)

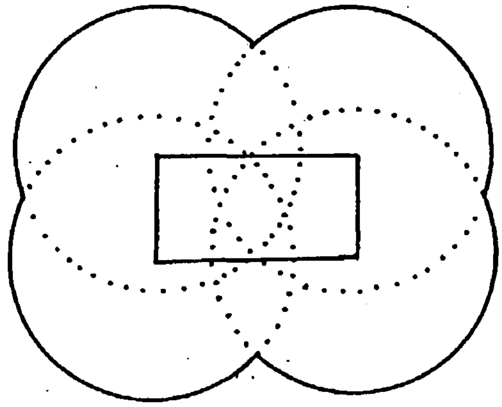

Fig.3, d)

$$F = 4u^2\{\pi - \arccos(a/2u) - \arccos(b/2u) + (a/2u)(1 - a^2/4u^2)^{1/2} + (b/2u)(1 - b^2/4u^2)^{1/2}\} \quad \text{if} \quad b<2u<(a^2+b^2)^{1/2}$$

$$F = 2u^2\{(3/2)\pi - \arccos(a/2u) - \arccos(b/2u) + (a/2u)(1 - a^2/4u^2) + (b/2u)(1 - b^2/4u^2)^{1/2} + ab/2u^2\} \quad \text{if} \quad (a^2+b^2)^{1/2} < 2u .$$

c) If T is an <u>equilateral</u> <u>triangle</u> of side a, we have (Fig.4, a, b, c)

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$$F = 3\pi u^2 \qquad \text{if} \qquad u \leq a/2$$

$$F = 6u^2 \arcsin(a/2u) + 3au(1 - a^2/4u^2)^{1/2} \qquad \text{if } a/2 \leq u \leq a/\sqrt{3};$$

$$F = \pi u^2 + 3u^2 \arcsin(a/2u) + (\sqrt{3}/4)a^2 + (3/2)au(1 - a^2/4u^2)^{1/2}$$
$$\text{if } a/\sqrt{3} \leq u \ .$$

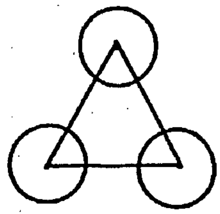

Fig.4, a)

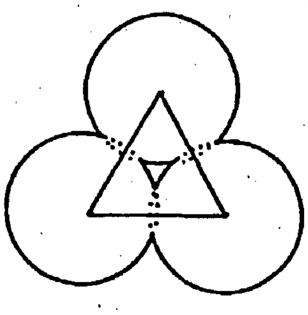

Fig.4, b)

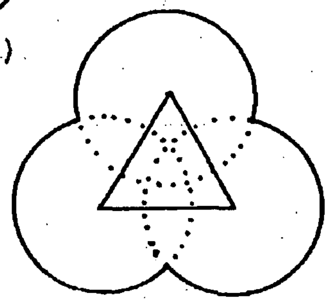

Fig.4, c)

2.3 <u>The distribution of the distance from a given point to the nearest graph</u>. Let $F^*(u)$ be the area of the set of points whose distance to T is $\leq u$. This function $F^*(u)$ is also a characteristic of T . Proceding as before we have that

The probability that the distance from a point Q chosen at random independently of the process of graphs, to the nearest graph exceeds u is $\exp(-\lambda F^*)$.

The function $F^*(u)$ must be calculated for each particular graph. For instance, by direct computation, it is easy to obtain the following results:

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a) If T is a line segment of length a, we have

$$F^*(u) = \pi u^2 + 2ua ;$$

b) If T is a rectangle of sides a, b such that $b \geqslant a$, we have (Fig.5, a, b):

$$F^*(u) = 4(a + b)u + \pi u^2 - 4u^2 , \quad \text{if} \quad 2u \leqslant a ;$$

$$F^*(u) = 2(a + b)u + \pi u^2 + ab , \quad \text{if} \quad a \leqslant 2u .$$

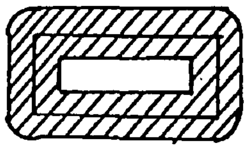

Fig.5, a)

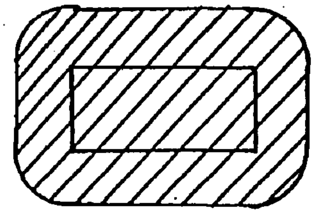

Fig.5, b)

c) If T is an equilateral triangle of side a, we have (Fig.6,a, b)

$$F^*(u) = \pi u^2 - 3\sqrt{3}\, u^2 + 6au , \quad \text{if} \quad u \leqslant a/2\sqrt{3} ;$$

$$F^*(u) = 3au + \pi u^2 + (\sqrt{3}/4)\, a^2 , \quad \text{if} \quad a/2\sqrt{3} \leqslant u .$$

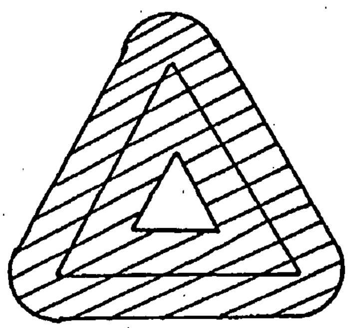

Fig.6 , a).

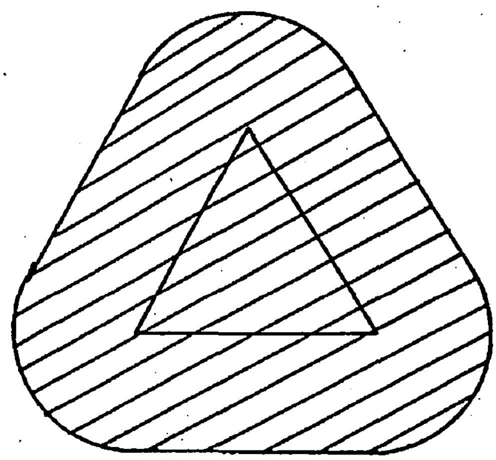

Fig.6, b)

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3. SECOND EXTENSION: SETS OF GEODESIC SEGMENTS ON SURFACES.

3.1 <u>Sets of geodesic segments which intersect a convex set</u>. The density for geodesic segments S on a surface Σ (riemannian space of dimension 2) is given by the same formula (1.2), where now dG stands for the density of geodesic lines of the surface (see [4]). A bounded set of points K on the surface is called convex if every geodesic has at most two points in common with the boundary ∂K, exceptimg geodesics which have an entire segment which belongs to ∂K. Let D be the diameter of K. If the surface Σ has closed geodesics and their minimal length is L_G, we will consider only segments whose maximal length s_m satisfies the inequality

$$D + s_m \leq L_G . \tag{3.1}$$

That means that $f(s) = 0$ for $s > s_m$.

With these conditions, the measure of the set of oriented geodesic segments which intersect a fixed convex set K of area F and perimeter L, is [4]

$$\int_{S \cap K \neq \emptyset} dS = 2\pi F + 2E(s)L . \tag{3.2}$$

and denoting by ν the number of extreme points (origin or end points) of the segments S intersecting K which are within K, we have

$$\int \nu \, dS = 4\pi F . \tag{3.3}$$

Let S be a segment which intersect K and let α denote the length of the part of S within K. Consider the integral

$$J_1 = \int dG \wedge dS \tag{3.4}$$

extended over all pairs (G,S) such that $G \cap S \in K$. If we first leave S fixed, we have $J_1 = 4 \int \alpha \, dS$ and if we first leave G fixed and σ denotes the length of the chord $G \cap K$, we have $J_1 = 4E(s) \int \sigma \, dG = 8\pi E(s)F$ (according to well known results of Integral Geometry, see for instance [5]). Hence, we have

$$\int \alpha \, dS = 2\pi E(s)F . \tag{3.5}$$

Consider now two geodesic segments S_1, S_2 which intersect K. Let n_{12} denote the function which is equal to 1 if $S_1 \cap S_2 \in K$ and is equal to zero otherwise. Consider the integral

$$J_2 = \int n_{12}\, dS_1 \wedge dS_2 \tag{3.6}$$

extended over all pairs such that $S_1 \cap S_2 \neq \emptyset$ $(i=1,2)$. If α_1 denotes the length of the intersection $S_1 \cap K$, integrating first with respect to S_2 we have $J_2 = 4E(s_2)\int \alpha_1 dS_1$, and according to (3.5) we have

$$J_2 = 8\pi F\, E(s_1)E(s_2)\ . \tag{3.7}$$

From (3.2), (3.3), (3.5) and (3.7) we deduce:

a) The mean value $E(\nu)$ of the number of extreme points (origin or end point) within K of n geodesic segments on a surface Σ which intersect at random a convex set K of Σ, is

$$E(\nu) = \frac{2\pi nF}{\pi F + E(s)L}\ . \tag{3.8}$$

b) For n geodesic segments S_i $(i=1,2,\ldots,n)$ chosen at random on the surface Σ, which intersect a convex set K, the mean value of the sum of the lengths α_i within K, is

$$E(\Sigma\alpha_i) = \pi F \sum_{1}^{n} \frac{E(s_i)}{\pi F + E(s_i)L}\ . \tag{3.9}$$

If all segments have the same mean length $E(s)$, we have

$$E(\Sigma\alpha_i) = \frac{n\pi F E(s)}{\pi F + E(s)L}. \tag{3.10}$$

c) For n geodesic segments S_i chosen independently at random on the surface Σ, which intersect a convex set K, the mean value of the number N of segment-segment crossings within K, is

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$$E(N) = 2\pi F \sum_{i<j} \frac{E(s_i)E(s_j)}{(\pi F + E(s_i)L)(\pi F + E(s_j)L)} . \tag{3.11}$$

If all segments have the same mean length $E(s)$, we have

$$E(N) = \frac{n(n-1)\pi F(E(s))^2}{(\pi F + E(s)L)^2} . \tag{3.12}$$

3.2 Sets of segments on the unit sphere. The measure of all geodesic segments on the unit sphere has a finite value, namely

$$\int_{Total} dS = \int f(s)\, ds \wedge dP \wedge d\phi = 8\pi^2 . \tag{3.13}$$

Hence, from the results above we can state:

a) The mean value of the number of extreme points (origin or end points) of n random segments chosen independently at random on the unit sphere, which lie within a convex set K , is

$$E(\nu) = \frac{nF}{2\pi} . \tag{3.14}$$

b) Let K be a convex set of diameter D on the unit sphere. Consider a random segment whose maximal length satisfies the inequality $D+s_m \leqslant 2\pi$. Then, the probability that $S \cap K \neq \emptyset$, is

$$p(S \cap K \neq \emptyset) = \frac{\pi F + E(s)L}{4\pi^2} . \tag{3.15}$$

c) The mean value of the total length within K of n random segments S_i chosen independently on the unit sphere, is

$$E(\Sigma\sigma_i) = (4\pi)^{-1} F \sum_{1}^{n} E(s_i) \tag{3.16}$$

If all segments have the same mean length $E(s)$, we have

$$(3.17)\qquad E(\Sigma\alpha_i) = \frac{nE(s)F}{4\pi} .$$

d) The mean number of segment-segment crossings within K of n segments placed independently at random on the sphere is

$$(3.18)\qquad E(N) = (8\pi^3)^{-1} F \sum_{i<j} E(s_i)E(s_j) .$$

In particular, if all segments have the same mean length $E(s)$, we have

$$(3.19)\qquad E(N) = \frac{n(n-1)(E(s))^2F}{16\pi^3} .$$

3.3 <u>Sets of segments on the hyperbolic plane.</u> From (3.2), (3.5) and (3.6) we deduce:

a) If K is a convex set interior to a convex set K_0 on a given surface Σ, the probability that a random segment intersecting K_0, also intersects K, is

$$(3.20)\qquad p = \frac{\pi F + E(s)L}{\pi F_0 + E(s)L_0} .$$

b) If we consider n random segments of the same mean length $E(s)$ which intersect K_0, the mean value of the total length of their intersection with K, is

$$(3.21)\qquad E(\Sigma\alpha_i) = \frac{n\pi E(s)F}{\pi F_0 + E(s)L_0} ,$$

and the mean value of the number N of segment-segment crossings within K, is

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$$E(N) = \frac{n(n-1)\pi(E(s))^2F}{(\pi F_0 + E(s)L_0)^2} \quad . \tag{3.22}$$

c) From (3.20) it follows that if there are chosen at random n segments which intersect K_0, the probability that exactly m of them intersect K, is

$$p_m = \binom{n}{m}\left(\frac{\pi F + E(s)L}{\pi F_0 + E(s)L_0}\right)^m \left(1 - \frac{\pi F + E(s)L}{\pi F_0 + E(s)L_0}\right)^{n-m} . \tag{3.23}$$

Assume that Σ is an unbounded surface of infinite area and let K_0 expand to the whole surface at the same time that $n \to \infty$ in such a way that

$$\frac{n}{F_0} \to \lambda \quad \text{(positive constant).} \tag{3.24}$$

Assuming moreover that under these conditions we have

$$\frac{L_0}{F_0} \to \kappa \quad , \tag{3.25}$$

then p_m tends to the limit

$$p_m = \frac{(\lambda H)^m}{m!}\exp(-\lambda H) \quad , \qquad H = \frac{\pi F + E(s)L}{\pi + \kappa E(s)} \quad . \tag{3.26}$$

It is known that $\kappa=0$ for the euclidean plane and $\kappa=1$ for the hyperbolic plane [6] . The obtained process is called an homogeneous Poisson segment process of intensity λ on Σ .

According to (3.26) the mean number of segments intersected by a convex set K placed at random on the surface is λH .

Using (3.21) and (3.22) we get that the mean value of the total length within K of segments that intersect with K is

$$E(\Sigma\sigma_i) = \frac{\lambda\pi E(s)F}{\pi + \kappa E(s)} \tag{3.27}$$

and the number of segment-segment crossings within K , is

$$(3.28)\qquad E(N) = \frac{\lambda^2\pi(E(s))^2F}{(\pi + \kappa E(s))^2} .$$

For the euclidean plane, $\kappa=0$, these results are the work of Parker and Cowan [3] . For the hyperbolic plane we must put $\kappa=1$.

d) Comsider the hyperbolic plane, $\kappa=1$. If D_r denotes the distance from a point Q chosen at random independently of the process, to the nearest r-th line segment, the probability that $D_r > u$ is equal to the probability that a disc of radius u placed at random on the plane intersect no more than $r-1$ line segments, that is

$$(3.29)\qquad p(D_r>u) = \sum_{m=0}^{r-1} \frac{(\lambda H)^m}{m!} \exp(-\lambda H)$$

where H is given by (3.26) with

$$(3.30)\qquad \kappa = 1 , \quad F = 2\pi(\cosh u - 1) , \quad L = 2\pi \sinh u .$$

For $r=1$ we have the probability that the distance from a point Q chosen at random on the hyperbolic plane to the nearest line segment is greater than u . Thus, the probability density function for the distances from Q to the nearest line segment is

$$(3.31)\qquad 2\lambda\pi(\pi + E(s))^{-1}(\pi \sinh u + E(s)\cosh u) \exp(-\lambda H)$$

with

$$(3.32)\qquad H = \frac{2\pi}{\pi + E(s)}\left(\pi(\cosh u - 1) + E(s) \sinh u\right) .$$

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