ANNALS OF MATHEMATICS
Vol. 47, No. 3, July, 1946

CONVEX REGIONS ON THE n-DIMENSIONAL SPHERICAL SURFACE

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(Received October 29, 1945)

1. Introduction

Let K be any bounded subset of n-dimensional euclidean space. If D is the diameter of K and R the radius of the smallest spherical surface enclosing K, it is known that the following relation holds:

$$R \leqq \left(\frac{n}{2n+2}\right)^{\frac{1}{2}} D. \tag{1.1}$$

This result was obtained by H. W. E. Jung [6], [7]; for bibliography until 1934 see Bonnesen-Fenchel [3, p. 78]. More recent proofs have been given by W. Süss [11] and L. M. Blumenthal-G. E. Wahlin [2].

If K is now a convex set of n-dimensional euclidean space, the "breadth" B of K is defined as the minimum distance of two parallel supporting hyperplanes of K. Let r be the radius of the greatest spherical surface which is contained in K. As a kind of dual of Jung's theorem (1.1) are known the relations

$$r \geqq \begin{cases} \dfrac{(n+2)^{\frac{1}{2}}}{2n+2} B & \text{for } n \text{ even} \\ \frac{1}{2}n^{-\frac{1}{2}} B & \text{for } n \text{ odd.} \end{cases} \tag{1.2}$$

For $n = 2$ this theorem was proved by W. Blaschke [1]; for any n by Steinhagen [10]; for bibliography until 1934 see Bonnesen-Fenchel [3, p. 79]. Another proof was given by H. Gericke [4].

The purpose of the present paper is to give a generalization of inequalities (1.1) and (1.2) to sets on the n-dimensional spherical surface. Whilst in the euclidean case it is necessary to give an independent proof for each inequality (1.1), (1.2) it will be enough to prove in the spherical case the generalization of Jung's inequality (1.1), because the generalization of the inequalities (1.2) will then follow by duality.

As an application of these results we obtain two theorems (Theorem 3 and 4) referring to convex regions on the n-dimensional spherical surface, the last of which generalizes a known theorem [8] of Robinson.

For $n = 2$ a geometrical proof of the results contained in this paper has been given by the author in [9].

2. Spherical Simplexes on $S_{n,1}$

An n-dimensional spherical surface $S_{n,1}$ is the "surface" of an $(n + 1)$-dimensional sphere of unit radius in the $(n + 1)$-dimensional euclidean space.

Let T_n be an equilateral spherical simplex on $S_{n,1}$, that is, the spherical simplex determined by $n + 1$ points $a_1, a_2, a_3, \cdots, a_{n+1}$ of $S_{n,1}$ whose mutual distances measured on $S_{n,1}$ have the constant value l (l = edge of T_n). If a_i also represents the unit vectors with the origin at the center of $S_{n,1}$ and with the end points a_i, we have

$$a_i^2 = 1, \qquad a_i a_j = \cos l. \tag{2.1}$$

The circumscribed sphere of T_n considered as an n-dimensional sphere of $S_{n,1}$ has a spherical center c, which is a point of $S_{n,1}$ and a spherical radius R, so that

$$c^2 = 1, \qquad ca_i = \cos R. \tag{2.2}$$

In order to calculate the value of R as a function of the edge l it suffices to observe that we can put

$$c = \sum_1^{n+1} \lambda_i a_i, \qquad \lambda_i > 0.$$

Since T_n is an equilateral simplex, we have $\lambda_i = \lambda$ = constant, and from (2.1) and (2.2), also

$$c^2 = (n + 1)\lambda^2 + n(n + 1)\lambda^2 \cos l = 1, \qquad ca_i = \cos R = \lambda + n\lambda \cos l$$

whence

$$\cos R = \left(\frac{1 + n \cos l}{n + 1}\right)^{\frac{1}{2}}. \tag{2.3}$$

We thus get the relation $\cos l \leqq -1/n$, which holds for any equilateral spherical simplex of $S_{n,1}$.

We wish now to calculate the spherical diameter of T_n. Let x, y be the end-points of a diameter of T_n. Let $a_1, a_2, \cdots a_\nu$ $(1 \leqq \nu \leqq n + 1)$ be the vertices of the simplex $T_{\nu-1}$ of minimal dimension whose vertices are among those of T_n and which contains the point x. The end-point y cannot be a point of $T_{\nu-1}$ because in this case x and y would be points of the boundary of $T_{\nu-1}$ and x would be contained in a simplex of dimension $<\nu$. Consequently y is a point which belongs to the simplex $T_{n-\nu}$ whose vertices are $a_{\nu+1}, a_{\nu+2}, \cdots, a_{n+1}$. Hence we have

$$x = \sum_1^{\nu} \lambda_i a_i, \qquad y = \sum_{\nu+1}^{n+1} \lambda_i a_i, \qquad \lambda_i \geqq 0 \tag{2.4}$$

with

$$\begin{aligned} x^2 &= \sum_1^{\nu} \lambda_i^2 + 2 \sum_{i,j=1}^{\nu}{}' \lambda_i \lambda_j \cos l = 1 \\ y^2 &= \sum_{\nu+1}^{n+1} \lambda_i^2 + 2 \sum_{i,j=\nu+1}^{n+1}{}' \lambda_i \lambda_j \cos l = 1 \end{aligned} \tag{2.5}$$

where $\sum'$ denotes a summation with $i = j$ excluded.

The spherical distance φ from x to y is given by

$$\cos\varphi = xy = \left(\sum_1^\nu \lambda_i\right)\left(\sum_{\nu+1}^{n+1} \lambda_i\right)\cos l. \tag{2.6}$$

To find the maximal value of φ we consider two cases:

a) $l \leqq \pi/2$, $\cos l \geqq 0$. By (2.5) we have

$$\begin{aligned} \left(\sum_1^\nu \lambda_i\right)^2 &= \sum_1^\nu \lambda_i^2 + 2\sum_{i,j=1}^{\nu}{}' \lambda_i\lambda_j = 1 + 2\sum_{i,j=1}^{\nu}{}' \lambda_i\lambda_j(1 - \cos l) \\ \left(\sum_{\nu+1}^{n+1} \lambda_i\right)^2 &= \sum_{\nu+1}^{n+1} \lambda_i^2 + 2\sum_{i,j=\nu+1}^{n+1}{}' \lambda_i\lambda_j = 1 + 2\sum_{i,j=\nu+1}^{n+1}{}' \lambda_i\lambda_j(1 - \cos l). \end{aligned} \tag{2.7}$$

The maximum value of φ corresponds to the minimum value of $\cos\varphi$, or according to (2.6) and (2.7) to $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \cdots = \lambda_\nu = 0$, $\lambda_{\nu+1} = 1$, $\lambda_{\nu+2} = \lambda_{\nu+3} = \cdots = \lambda_{n+1} = 0$. Consequently the spherical diameter of T_n equals the edge l

b) $l > \pi/2$, $\cos l < 0$. By (2.6) the maximum value of φ corresponds to the maximum of the product $(\sum_1^\nu \lambda_i)$ $(\sum_{\nu+1}^{n+1} \lambda_i)$ with the relations (2.7). Taking into account the inequality

$$\lambda_i\lambda_j \leqq \tfrac{1}{2}(\lambda_i^2 + \lambda_j^2) \tag{2.8}$$

and $\cos l < 0$, we deduce from (2.5)

$$1 \geqq \sum_1^\nu \lambda_i^2 + \sum_{i,j=1}^{\nu}{}' (\lambda_i^2 + \lambda_j^2)\cos l = \sum_1^\nu \lambda_i^2(1 + (\nu - 1)\cos l). \tag{2.9}$$

Moreover, since

$$\left(\sum_1^\nu \lambda_i\right)^2 \leqq \nu \sum_1^\nu \lambda_i^2 \tag{2.10}$$

we deduce from (2.9)

$$\left(\sum_1^\nu \lambda_i\right)^2 \leqq \frac{\nu}{1 + (\nu - 1)\cos l}. \tag{2.11}$$

Analogously we get

$$\left(\sum_{\nu+1}^{n+1} \lambda_i\right)^2 \leqq \frac{n - \nu + 1}{1 + (n - \nu)\cos l}. \tag{2.12}$$

From (2.6), (2.11) and (2.12) and $\cos l < 0$ the inequality

$$\cos\varphi \geqq \frac{\nu^{\frac{1}{2}}(n - \nu + 1)^{\frac{1}{2}}\cos l}{(1 + (\nu - 1)\cos l)^{\frac{1}{2}}(1 + (n - \nu)\cos l)^{\frac{1}{2}}} \tag{2.13}$$

follows. There is equality only when $\lambda_1 = \lambda_2 = \cdots = \lambda_\nu$ and $\lambda_{\nu+1} = \lambda_{\nu+2} = \cdots = \lambda_{n+1}$.

By considering the right hand member of (2.13) as a function of the continuous variable ν and equating to 0 its derivative, we find

$$(n - 2\nu + 1)\,(1 + (n - 1)\cos l - n \cos^2 l) = 0.$$

Except the limit cases $\cos l = 1$, $\cos l = -1/n$, the value of ν for which the right hand member of (2.13) is a minimum corresponds to the integral solution of the equation $n - 2\nu + 1 = 0$. Hence we have

$$\cos \varphi \geqq \frac{(n + 1)\cos l}{2 + (n - 1)\cos l} \qquad \text{for } n \text{ odd} \tag{2.14}$$

$$\cos \varphi \geqq \frac{(n(n + 2))^{\frac{1}{2}} \cos l}{(2 + (n - 2)\cos l)^{\frac{1}{2}}(2 + n \cos l)^{\frac{1}{2}}} \qquad \text{for } n \text{ even.} \tag{2.15}$$

The right hand member of (2.14) is the spherical distance between two opposite spherical simplexes $T_{\frac{1}{2}(n-1)}$ of T_n. The right hand member of (2.15) is the spherical distance from a $T_{n/2}$ to the opposite $T_{n/2-1}$ of T_n.

Summing up our conclusions we have

LEMMA 1. *The spherical diameter D_0 of an equilateral spherical simplex T_r of edge l has the following values:*

a) *If $l \leqq \pi/2$, $D_0 = l$, that is, in virtue of* (2.3)

$$\cos D_0 = \frac{1}{n}((n + 1)\cos^2 R - 1). \tag{2.16}$$

b) *If $l > \pi/2$ (and always $\cos l \geqq -1/n$), that is, for $\cos R \leqq (n + 1)^{-\frac{1}{2}}$ it is*

$$\cos D_0 = \frac{(n + 1)\cos l}{2 + (n - 1)\cos l} \qquad \textit{for } n \textit{ odd} \tag{2.17}$$

$$\cos D_0 = \frac{(n(n + 2))^{\frac{1}{2}} \cos l}{(2 + (n - 2)\cos l)^{\frac{1}{2}}(2 + n \cos l)^{\frac{1}{2}}} \qquad \textit{for } n \textit{ even} \tag{2.18}$$

or, according to (2.3)

$$\cos D_0 = \frac{(n + 1)\cos^2 R - 1}{1 + (n - 1)\cos^2 R} \qquad \textit{for } n \textit{ odd} \tag{2.19}$$

$$\cos D_0 = \frac{(n + 1)\cos^2 R - 1}{(1 + (n + 1)\cos^2 R)^{\frac{1}{2}}\left(1 + \frac{(n + 1)(n - 2)}{n + 2}\cos^2 R\right)^{\frac{1}{2}}} \tag{2.20}$$

for n even.

These formulas (2.19) and (2.20) will be needed in the sequel.

Let us consider now on $S_{n,1}$ a non equilateral simplex T'_n and let $b_1, b_2, b_3, \cdots, b_{n+1}$ be its vertices. Suppose that R is the spherical radius of the circumscribed sphere of T'_n and that the point c of $S_{n,1}$ is the spherical center of this circumscribed sphere. Let us suppose also that c belongs to T'_n. We shall prove the following

LEMMA 2. *Among the n-dimensional spherical simplexes on $S_{n,1}$ whose circumscribed sphere has the spherical radius R and which contain the spherical center of its circumscribed sphere, the equilateral simplex has a minimum diameter.*

PROOF. Let l be the edge of the equilateral spherical simplex T_n inscribed in the sphere of spherical radius R. We consider two cases:

1°. $l \leqq \pi/2$. In this case the spherical diameter of T_n is l. We shall prove that there exist at least one edge of T'_n of length greater than l. Since c is a point of T'_n we can write

$$c = \sum_1^{n+1} \mu_i b_i \quad \text{with} \quad \mu_i \geqq 0 \tag{2.21}$$

where

$$c^2 = \sum_1^{n+1} \mu_i^2 + 2 \sum_{i,j=1}^{n+1}{}' \mu_i \mu_j b_i b_j = 1. \tag{2.22}$$

We have also

$$\cos R = cb_k \quad \text{and} \quad \sum_1^{n+1} \mu_k \cos R = c \sum_1^{n+1} \mu_k b_k = c^2 = 1$$

whence

$$\cos R = \left(\sum_1^{n+1} \mu_k \right)^{-1}. \tag{2.23}$$

If all the edges of T'_n were smaller than or equal to l, we would have $b_i b_k \geqq \cos l$ since $l \leqq \pi/2$ and, since $\mu_i \geqq 0$,

$$\cos R = cb_k = \mu_k + \sum_{i=1}^{n+1}{}' \mu_i b_i b_k \geqq \mu_k + \sum_1^{n+1}{}' \mu_i \cos l.$$

since T'_n is not equilateral we get, by adding for $k = 1, 2, 3, \cdots, n+1$,

$$(n+1) \cos R > (1 + n \cos l) \sum_1^{n+1} \mu_k$$

and from (2.23)

$$\cos^2 R > \frac{1 + n \cos l}{n+1},$$

in contradiction to (2.3). This proves that at least one of the products $b_i b_k$ is smaller than $\cos l$ and consequently at least one of the edges of T'_n is greater than l.

2°. $l > \pi/2$. In this case the diameter D_0 of T_n is given by the formulas (2.19), (2.20). Let us consider two cases:

a) *n odd.* Let us put $m = \frac{1}{2}(n+1)$. We can write (2.21) in the form

$$c = \sum_1^{m} \mu_i b_i + \sum_{m+1}^{n+1} \mu_i b_i, \qquad \mu_i \geqq 0. \tag{2.24}$$

If δ is the spherical distance between the points $\sum_1^m \mu_i b_i / |\sum_1^m \mu_i b_i|$ and $\sum_{m+1}^{n+1} \mu_i b_i / |\sum_{m+1}^{n+1} \mu_i b_i|$ which belong to $S_{n,1}$, from (2.24) we deduce

$$c \sum_1^m \mu_i b_i = \sum_1^m \mu_i \cos R = \left(\sum_1^m \mu_i^2 + 2 \sum_{i,j=1}^{m}{}' \mu_i \mu_j b_i b_j \right) \tag{2.25}$$
$$+ \left(\sum_1^m \mu_i^2 + 2 \sum_{i,j=1}^{m}{}' \mu_i \mu_j b_i b_j \right)^{\frac{1}{2}} \left(\sum_{m+1}^{n+1} \mu_i^2 + 2 \sum_{i,j=m+1}^{n+1}{}' \mu_i \mu_j b_i b_j \right)^{\frac{1}{2}} \cos \delta.$$

Since $l > \pi/2$, from (2.17) it follows that $D_0 > \pi/2$. If the distance δ were smaller than or equal to D_0, we would have $\cos \delta \geqq \cos D_0$, and from (2.25),

$$\sum_1^m \mu_i \cos R - \left(\sum_1^m \mu_i^2 + 2 \sum_{i,j=1}^{m}{}' \mu_i \mu_j b_i b_j \right) - \left(\sum_1^m \mu_i^2 + 2 \sum_{i,j=1}^{m}{}' \mu_i \mu_j b_i b_j \right)^{\frac{1}{2}} \tag{2.26}$$
$$\cdot \left(\sum_{m+1}^{n+1} \mu_i^2 + 2 \sum_{i,j=m+1}^{n+1}{}' \mu_i \mu_j b_i b_j \right)^{\frac{1}{2}} \cos D_0 \geqq 0.$$

Since $\cos D_0 < 0$, by applying the theorem of the arithmetic and geometric means, we obtain

$$\sum_1^m \mu_i \cos R - \left(\sum_1^m \mu_i^2 + 2 \sum_{i,j=1}^{m}{}' \mu_i \mu_j b_i b_j \right)$$
$$- \frac{1}{2} \left(\sum_1^{n+1} \mu_i^2 + 2 \sum_{i,j=1}^{m}{}' \mu_i \mu_j b_i b_j + 2 \sum_{i,j=m+1}^{n+1}{}' \mu_i \mu_j b_i b_j \right) \cos D_0 \geqq 0.$$

Writing this inequality for all the $\binom{n+1}{m}$ possible combinations of $\mu_1, \mu_2, \cdots, \mu_{n+1}$ taken m at a time and summing, we get, since T'_n is not equilateral and $c^2 = 1$,

$$\binom{n}{m-1} \sum_1^{n+1} \mu_i \cos R - \left[\binom{n}{m-1} \sum_1^{n+1} \mu_i^2 + \alpha \left(1 - \sum_1^{n+1} \mu_i^2 \right) \right] \tag{2.27}$$
$$- \frac{1}{2} \left[\binom{n+1}{m} \sum_1^{n+1} \mu_i^2 + 2\alpha \left(1 - \sum_1^{n+1} \mu_i^2 \right) \right] \cos D_0 > 0$$

where

$$\alpha = \frac{\binom{m}{2}\binom{n+1}{m}}{\binom{n+1}{2}} = \frac{n-1}{4n} \binom{n+1}{m}. \tag{2.28}$$

Taking into account (2.23) which is also valid for the present case, we have

$$\cos D_0 < \frac{2\left(\alpha - \binom{n}{m-1} \right) \left(\sum_1^{n+1} \mu_i^2 - 1 \right)}{\left(\binom{n+1}{m} - 2\alpha \right) \sum_1^{n+1} \mu_i^2 + 2\alpha}. \tag{2.29}$$

Furthermore

$$\frac{1}{\cos^2 R} = \sum_1^{n+1} \mu_i^2 + 2 \sum_{i,j=1}^{n+1}{}' \mu_i \mu_j$$

and since $2\mu_i\mu_j \leqq \mu_i^2 + \mu_j^2$, we have

$$\frac{1}{\cos^2 R} \leqq (n + 1) \sum_1^{n+1} \mu_i^2 . \tag{2.30}$$

By (2.28), (2.29) and (2.30) we obtain

$$\cos D_0 < \frac{(n + 1) \cos^2 R - 1}{1 + (n - 1) \cos^2 R} .$$

We have arrived at a contradiction with (2.19). This proves that the assumption that all the distances δ were smaller than or equal to D_0 is false. Consequently the simplex T'_n has not smaller diameter than the equilateral simplex T_n.

b) *n even.* Let us put $m = n/2$. We may proceed as above until the inequality (2.26). Then summing for all the $\binom{n+1}{m}$ possible combinations of μ_1, $\mu_2, \cdots, \mu_{n+1}$ taken m at a time and applying the known Cauchy's inequality (see [5, p. 16])

$$\sum \xi\eta \leqq \left(\sum \xi^2\right)^{\frac{1}{2}}\left(\sum \eta^2\right)^{\frac{1}{2}}$$

we obtain since T'_n is supposed non equilateral and $c^2 = 1$,

$$\begin{aligned} &\binom{n}{m-1} \sum_1^{n+1} \mu_i \cos R - \left[\binom{n}{m-1} \sum_1^{n+1} \mu_i^2 + \alpha\left(1 - \sum_1^{n+1} \mu_i^2\right)\right] \\ &\quad - \left[\binom{n}{m-1} \sum_1^{n+1} \mu_i^2 + \alpha\left(1 - \sum_1^{n+1} \mu_i^2\right)\right]^{\frac{1}{2}} \\ &\quad \cdot \left[\binom{n}{m} \sum_1^{n+1} \mu_i^2 + \alpha'\left(1 - \sum_1^{n+1} \mu_i^2\right)\right]^{\frac{1}{2}} \cos D_0 > 0 \end{aligned} \tag{2.31}$$

where

$$\alpha = \frac{n-2}{2n}\binom{n}{m-1}, \qquad \alpha' = \frac{1}{2}\binom{n}{m}.$$

From (2.31) and (2.23) it follows that

$$\cos D_0 < \frac{1 - \sum_1^{n+1} \mu_i^2}{\left(\frac{n-2}{n+2} + \sum_1^{n+1} \mu_i^2\right)^{\frac{1}{2}}\left(1 + \sum_1^{n+1} \mu_i^2\right)^{\frac{1}{2}}}$$

and taking into account (2.30) we get

$$\cos D_0 < \frac{(n + 1) \cos^2 R - 1}{\left(1 + \frac{(n-2)(n+1)}{n+2} \cos^2 R\right)^{\frac{1}{2}} (1 + (n + 1) \cos^2 R)^{\frac{1}{2}}}$$

in contradiction to (2.20). Consequently the assumption that all the distances

δ were smaller than D_0 was false. Hence the simplex T'_n does not have a smaller diameter than the equilateral simplex T_n.

This completes the proof of the Lemma 2.

3. Circumscribed Sphere to Sets on the n-Dimensional Sphere

Let us consider sets of points on $S_{n,1}$, that is, on the surface of the sphere of unit radius in the $(n + 1)$-dimensional euclidean space. We consider only sets which lie entirely in a hemisphere, hence, its spherical diameter D is always $\leqq \pi$.

Given a set K on $S_{n,1}$, the smallest sphere on $S_{n,1}$ enclosing K is called the "circumscribed sphere" to K; let R be its spherical radius. We have $R \leqq \pi/2$.

Our purpose is to give an inequality between R and D valid for any set K, which will be the generalization to $S_{n,1}$ of the Jung's inequality (1.1). That is to say, given R we wish to find the minimum value of D. For our purpose we can assume without loss of generality that K is a closed set.

Following the same way as in euclidean case (see Bonnesen-Fenchel [3, p. 77]) it is seen that the circumscribed sphere to K contains points of K which form a set K' whose spherical convex cover (*konvexe Hülle*, [3, p. 5]) contains the spherical center c of the circumscribed sphere. Hence we can choose points of K' forming the vertices of a spherical simplex T' whose diameter is not greater than the diameter of K, which contains the center c and has the same circumscribed sphere as T. Consequently to find the minimal value of D it suffices to consider only simplexes with the same circumscribed sphere of spherical radius R which contain the center of this sphere.

It can happen that the dimension of T' be smaller than n, but as the left hand sides of (2.16), (2.19) and (2.20) increase with n, we have, in virtue of Lemmas 1 and 2:

THEOREM 1.—*For any set K on the surface $S_{n,1}$ of the $(n + 1)$-dimensional euclidean sphere of unit radius which lie on an hemisphere, the spherical diameter D of K and the spherical radius R of its circumscribed sphere satisfy the following relations*

1°. *If* $\cos R \geqq (n + 1)^{-\frac{1}{2}}$ *it is*

$$\cos 2R \leqq \cos D \leqq \frac{(n + 1) \cos^2 R - 1}{n} \tag{3.1}$$

2°. *If* $0 \leqq \cos R \leqq (n + 1)^{-\frac{1}{2}}$ *it is*

$$\cos 2R \leqq \cos D \leqq \frac{(n + 1) \cos^2 R - 1}{1 + (n - 1) \cos^2 R} \quad \textit{for } n \textit{ odd} \tag{3.2}$$

$$\cos 2R \leqq \cos D \leqq \frac{(n + 1) \cos^2 R - 1}{(1 + (n + 1) \cos^2 R)^{\frac{1}{2}}\left(1 + \frac{(n + 1)(n - 2)}{n + 2} \cos^2 R\right)^{\frac{1}{2}}} \tag{3.3}$$

for n even.

4. Inscribed Sphere in a Convex Set on the n-Dimensional Sphere

A set K on the n-dimensional spherical surface of unit radius is said to be convex when: 1. It lies in an hemisphere of $S_{n,1}$. 2. Any great circle arc of $S_{n,1}$ whose end points lie in K, lies entirely in K.

A closed convex n-dimensional spherical set K is called a "convex spherical region".

Two great spheres of $S_{n,1}$ (generalization of the great circles of the sphere in ordinary euclidean space) divide $S_{n,1}$ into four "lunes". Let B be the angle of the smallest lune containing the convex spherical region K. We shall call B the "spherical breadth" of K.

The greatest sphere on $S_{n,1}$ which is enclosed in K is called the "inscribed sphere" of K; let r be its spherical radius.

The diametral hyperplanes perpendicular to the radii of $S_{n,1}$ which projects the surface of the convex region K from the center of $S_{n,1}$ envelop a cone whose intersection with $S_{n,1}$ is the surface of a new convex region K^*. We shall call K^* the dual region of K.

The spherical radius r of the inscribed sphere and the spherical breadth B of K are connected with the spherical radius R^* of the circumscribed sphere and the spherical diameter D^* of K^* by the relations

$$D^* + B = \pi, \qquad R^* + r = \pi/2.$$

Consequently, transforming by duality the Theorem 1, we obtain:

THEOREM 2. *For any convex spherical region K on the surface $S_{n,1}$ of the $(n+1)$-dimensional euclidean sphere of unit radius, the spherical breadth B and the spherical radius r of its inscribed sphere satisfy the following relations:*

1°. *If* $\sin r \geqq (n+1)^{-\frac{1}{2}}$ *it is*

$$\cos 2r \geqq \cos B \geqq \frac{1-(n+1)\sin^2 r}{n} \tag{4.1}$$

2°. *If* $0 \leqq \sin r \leqq (n+1)^{-\frac{1}{2}}$ *it is*

$$\cos 2r \geqq \cos B \geqq \frac{1-(n+1)\sin^2 r}{1+(n-1)\sin^2 r} \quad \textit{for } n \textit{ odd} \tag{4.2}$$

$$\cos 2r \geqq \cos B \geqq \frac{1-(n+1)\sin^2 r}{(1+(n+1)\sin^2 r)^{\frac{1}{2}}\left(1+\frac{(n+1)(n-2)}{n+2}\sin^2 r\right)^{\frac{1}{2}}} \tag{4.3}$$

for n even.

5. Passage to the Case of the n-Dimensional Euclidean Space

The formulas (1.1) and (1.2) for the euclidean space must result as a limit case of the preceding Theorems 1 and 2 when the radius of the spherical surface $S_{n,1}$ increases indefinitely.

If we now consider the n-dimensional spherical surface $S_{n,a}$ of radius a, the values R, D, r, B which are in the formulas of the Theorems 1 and 2, must be replaced by R/a, D/a, r/a, B/a.

Let us first consider Theorem 1. In order that the convex spherical region K tends to a bounded convex region of the n-dimensional euclidean space as $a \to \infty$, we must take the case $\cos R \geqq (n+1)^{-\frac{1}{2}}$ and we obtain by (3.1)

$$\cos (D/a) \leqq \frac{(n+1)\cos^2 (R/a) - 1}{n}$$

whence, by great values of a

$$1 - \frac{D^2}{2a^2} + \cdots \leqq 1 - \frac{n+1}{n}\frac{R^2}{a^2} + \cdots .$$

Simplifying and multiplying both sides by a^2 and making $a \to \infty$ we obtain the inequality (1.1).

Let us now consider Theorem 2. In order that the convex spherical region K tends to a bounded convex region of the n-dimensional euclidean space as $a \to \infty$, we must take the case $\sin r \leqq (n+1)^{-\frac{1}{2}}$ and we obtain, by (4.2) and (4.3)

$$\cos (B/a) \geqq \frac{1 - (n+1)\sin^2 (r/a)}{1 + (n-1)\sin^2 (r/a)} \quad \text{for } n \text{ odd}$$

$$\cos (B/a) \geqq \frac{1 - (n-1)\sin^2 (r/a)}{(1 + (n+1)\sin^2 (r/a))^{\frac{1}{2}}\left(1 + \frac{(n+1)(n-2)}{n+2}\sin^2 (r/a)\right)^{\frac{1}{2}}}$$

for n even

whence, for large values of a,

$$1 - \frac{B^2}{2a^2} + \cdots \geqq 1 - 2n\frac{r^2}{a^2} + \cdots \quad \text{for } n \text{ odd}$$

$$1 - \frac{B^2}{2a^2} + \cdots \geqq 1 - \frac{2(n+1)^2}{n+2}\frac{r^2}{a^2} + \cdots \quad \text{for } n \text{ even.}$$

Simplifying and multiplying both sides by a^2 and making $a \to \infty$ we obtain the inequalities (1.2).

6. Two Theorems on Convex Regions on the n-Dimensional Spherical Surface

Let K be a convex region with spherical diameter D on $S_{n,1}$. Clearly there is always on $S_{n,1}$ an $(n-1)$-dimensional spherical surface of spherical radius $R_1 = \frac{1}{2}(\pi - D)$ which intersect both K and its symmetrical region with respect the center of $S_{n,1}$.

For $R_1 = R$ we shall have the minimum value of R for which, for any K, there exist an $(n-1)$-dimensional sphere of spherical radius R on $S_{n,1}$ which either encloses K in its interior or intersects both K and its symmetrical region with respect the center of $S_{n,1}$.

For $\cos R \leqq (n + 1)^{-\frac{1}{2}}$ by (3.2) and (3.3) it is $\cos D \leqq 0$, hence $D \geqq \pi/2$, $R_1 \leqq \pi/4$ and $\cos R_1 \geqq 2^{-\frac{1}{2}}$; consequently $R_1 \neq R$. To be $R = R_1$ we may therefore assume that $\cos R \geqq (n + 1)^{-\frac{1}{2}}$ and then introducing in (3.1) the value $D = \pi - 2R$ we have

$$-\cos 2R = \frac{(n + 1) \cos^2 R - 1}{n}$$

whence

$$\tan R = \left(\frac{2n}{n + 1}\right)^{\frac{1}{2}}. \tag{6.1}$$

For the equilateral spherical simplex on $S_{n,1}$ inscribed in the $(n - 1)$-dimensional spherical surface of spherical radius R given by (6.1), it is $D = l$ (l = edge of the simplex) and $\pi - l = 2R$. This proves that the value of R given by (6.1) cannot be diminished.

We have established the following theorem

THEOREM 3. *Let K be a convex spherical region of $S_{n,1}$. There is always an $(n - 1)$-dimensional spherical surface of $S_{n,1}$ with spherical radius R given by* (6.1) *such that it either incloses K in its interior or intersects both K and its symmetrical region with respect the center of $S_{n,1}$. The value of R given by* (6.1) *cannot be diminished.*

By duality this theorem can be announced

THEOREM 4. *Let K be a convex region of $S_{n,1}$. There is always an $(n - 1)$-dimensional spherical surface of $S_{n,1}$ with spherical radius r given by*

$$\tan r = \left(\frac{n + 1}{2n}\right)^{\frac{1}{2}} \tag{6.2}$$

such that it is either enclosed in K or has neither any point in common with K nor with the symmetrical region of K with respect the center of $S_{n,1}$. The value of r given by (6.2) *cannot be increased.*

For $n = 2$ this theorem has been obtained by Robinson [8].

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