# 36. On Einstein's Unified Field Theory

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**Abstract.** First, we obtain the most general tensor $T_{ij}$ of rank 2 in a space with a nonsymmetric affine connection which satisfies the following conditions: (a) it depends on the connection and its partial derivatives of the first ~~and second~~ order; (b) as a function of the connection it is, at most, of second degree. With $T_{ij}$ and a nonsymmetric tensor $g_{ij}$ we form the classical scalar density $T_{ij}g^{ij}|g|^{1/2}$ and deduce the field equations from the corresponding variational principle. The equations are complicated, but they may be expressed in a simple way by introducing two new connections. Second, we investigate the necessary assumptions on $T_{ij}$ in order that the field equations take the most simple form compatible with the nature of the problem. This form turns out to be that of the weak system of Einsten's unified theory of 1950. In particular, the simplifications induced by the assumptions of $\lambda$-invariance and pseudo-Hermitian symmetry are investigated.

## 1. Introduction

The starting point of the general theory of relativity is the assumption that the space-time is a four-dimensional Riemannian manifold. From this fundamental hypothesis, the field equations $G_{ij} = 0$ (outside the mass points) are determined by the "conditions of simplicity" contained in the following theorem of E. Cartan: $G_{ij}$ is the unique symmetric tensor of second rank such that: (a) it depends only on the fundamental tensor $g_{ij}$ and its first and second derivatives, and it is linear in the second derivatives; (b) it satisfies the conservation equations $G^{ij}{}_{;i} = 0$.[1,*]

No similar theorem exists for Einstein's unified theory of 1950 and its successive modifications.[2] The basic hypothesis is now that the space-time is a manifold with a nonsymmetric affine connection $\Gamma_{ij}{}^{m}$, and a nonsymmetric tensor $g_{ij}$, connected by the field equations, which are chosen in base to different criterions, all plausible and all disputable. This lack of determination may be the underlying reason for its little success to date.

In order to guarantee the compatibility of the field equations, it seems natural to restrict the possible systems to those which are derivable from a variational principle. It also seems natural to choose as the integrand of the variational principle a scalar density built out of the $g_{ij}$, $\Gamma_{ij}{}^{m}$ and their first derivatives. There are however, many possible scalar densities which satisfy these conditions. In order to reduce the number of such possibilities, Einstein imposes some extra conditions on the field equations, for instance, the "pseudo-Hermitian symmetry"[2] or the $\lambda$-invariance.[3]

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* See reference 6, pp. 9-10.

Our purpose is to start from the most general tensor $T_{ij}$ (12) of rank 2, which satisfies certain conditions of simplicity, and then to obtain the field equations which derive from the variational principle (16). The resulting equations are complicated, and we discuss the necessary assumptions on $T_{ij}$, in order that the field equations take the most simple form compatible with the nature of the problem. This form turns out to be that of the weak system of Einstein (40), where $T_{ij}$ has the general form (12), with the conditions of (35) and (38).

## 2. Notations and Classical Field Equations

Let $\Gamma_{ih}{}^{m}$ be an affine connection, and let

$$\Delta_{ih}{}^{m} = \tfrac{1}{2}(\Gamma_{ih}{}^{m} + \Gamma_{hi}{}^{m}), \qquad S_{ih}{}^{m} = \tfrac{1}{2}(\Gamma_{ih}{}^{m} - \Gamma_{hi}{}^{m}) \tag{1}$$

be its symmetric and skewsymmetric parts; $\Delta_{ih}{}^{m}$ is a connection and $S_{ih}{}^{m}$ is a tensor (tensor of torsion). Following Einstein, we set

$$S_{i} = S_{im}{}^{m}. \tag{2}$$

We denote by a comma the ordinary partial derivative, and by a semicolon the covariant derivative, with respect to the connection $\Gamma_{ih}{}^{m}$. The Ricci tensor is

$$R_{ih} = \Gamma_{ih,m}^{m} - \Gamma_{im,h}^{m} + \Gamma_{lm}^{m}\Gamma_{ih}{}^{l} - \Gamma_{lh}{}^{m}\Gamma_{im}^{l}. \tag{3}$$

Let $g_{ij}$ be a nonsymmetric tensor. If $g$ denotes the determinant of $g_{ij}$, assumed $\neq 0$, we introduce the density

$$G_{ih} = g_{ih}\sqrt{|g|} \tag{4}$$

which may be broken down into in its symmetric and skewsymmetric parts

$$H_{ih} = \tfrac{1}{2}(G_{ih} + G_{hi}), \qquad F_{ih} = \tfrac{1}{2}(G_{ih} - G_{hi}). \tag{5}$$

For any tensor or density of second rank, Einstein introduces the mixed covariant derivative obtained when one differentiates the first index with respect to $\Gamma_{ih}{}^{m}$, and the second index with respect to $\overline{\Gamma}_{ih}{}^{m} = \Gamma_{hi}{}^{m}$. We will denote this mixed covariant derivative by a vertical bar. For instance, we put

$$G^{ih}_{|s} = G^{ih}_{,s} + \Gamma^{i}_{ms}G^{ms} + \Gamma^{h}_{sm}G^{im} - \Delta^{m}_{sm}G^{ih}. \tag{6}$$

The set of equations that Einstein proposes for his unified field theory is

$$G^{ih}_{|s} = 0, \qquad F_{,i}{}^{ih} = 0, \qquad R_{(ih)} = 0,$$
$$R_{[ih],k} + R_{[hk],i} + R_{[ki],h} = 0, \tag{7}$$

where ( ) denotes the symmetric part and [ ] the skewsymmetric part of the Ricci tensor $R_{ih}$.[2,5,6,10] These equations result from the variational principle

$$\delta \int R_{ih} G^{ih} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = 0, \tag{8}$$

where $\Gamma_{ih}{}^m$ and $G^{ih}$ are to be varied independently of one another and their variations are chosen so that they vanish at the boundaries of integration.

In the general theory of relativity, the field equations are deduced from the same variational principle shown in eq. (8), with $g_{ij}$ as a unique variable, since the space-time is assumed to be a Riemannian space.[6] In this case the choice is justified since the Ricci tensor is the most simple tensor of rank 2 (besides $g_{ij}$). In the case of spaces with an affine connection, there is not such an unequivocal choice, since there are other tensors with analogous properties of simplicity. These tensors are those we want to consider in the next section.

## 3. The Field Equations

According to the general theorem of equivalence, the only independent tensors in the spaces with an affine connection are the tensor of torsion $S_{ih}{}^m$, the curvature tensor $R_{ihs}{}^m$ and their contractions and covariant derivatives.* The contractions of $R_{ihs}{}^m$ are the Ricci Tensor (3), and the tensor

$$R^m_{mih} = \Delta^m_{mi,h} - \Delta^m_{mh,i} + S_{h,i} - S_{i,h}. \tag{9}$$

Therefore, since

$$S_{i,h} - S_{h,i} = S_{i;h} - S_{h;i} + 2S_m S_{ih}{}^m, \tag{10}$$

we have the following theorem:

THEOREM 1: *In a space with an affine connection* $\Gamma_{ih}{}^m = \Delta_{ih}{}^m + S_{ih}{}^m$, *the only tensors of rank 2 which satisfy the conditions: (a) they depend on* $\Gamma_{ih}{}^m$ *and their partial derivatives of the first order; (b) as functions of* $\Gamma_{ih}{}^m$ *they are, at most, of second degree, are the following eight tensors:*

$$R_{ih}, \quad \Delta^m_{im,h} - \Delta^m_{hm,i}, \quad S_{ih;m}^m, \quad S_{i;h}, \quad S_{h;i},$$

$$S_{ir}{}^q S_{hq}^r, \quad S_i S_h, \quad S_m S_{ih}{}^m. \tag{11}$$

Therefore, the most general tensor of rank 2 which satisfies the conditions (a) and (b) is:

$$T_{ih} = \alpha R_{ih} + \beta(\Delta^m_{im,h} - \Delta^m_{hm,i}) + \gamma S_{ih;m}^m + \delta S_{ir}{}^q S_{hq}{}^r$$
$$+ \epsilon S_{i;h} + \varphi S_{h;i} + \mu S_m S_{ih}{}^m + \nu S_i S_h, \tag{12}$$

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* See reference 9, p. 204-205.

where $\alpha, \beta, \gamma, \ldots, \nu$ are arbitrary constants. As examples of such tensors, we have the following, which have been considered by various authors:

(a) The Einstein tensor,[2]

$$E_{ih} = -\tfrac{1}{2}(\Delta_{im,h}^{m} + \Delta_{hm,i}^{m}) + \Gamma_{ih,m}^{m} + \Gamma_{ih}{}^{l}\Delta_{lm}^{m} - \Gamma_{il}{}^{m}\Gamma_{mh}^{l} \quad (13)$$

which corresponds to

$$\alpha = 1, \beta = \tfrac{1}{2}, \epsilon = 1, \gamma = \delta = \varphi = \mu = \nu = 0.$$

(b) The tensors

$$^{(1)}R_{ij} = R_{ij} + \tfrac{2}{3}(S_{j,i} - S_{i,j})$$

$$^{(2)}R_{ij} = \Gamma_{ij,m}^{m} - \Delta_{im,j}^{m} + \Gamma_{ij}{}^{m}\Delta_{ml}^{l} - \Gamma_{im}^{l}\Gamma_{lj}{}^{m}$$

$$+ \tfrac{1}{3}(S_{j,i} - S_{i,j}) - \tfrac{1}{3}S_iS_j$$

$$^{(3)}R_{ij} = {}^{(2)}R_{ij} - \tfrac{1}{2}(\Delta_{jm,i}^{m} - \Delta_{im,j}^{m}), \quad (14)$$

which have been considered by Tonnelat,* and correspond respectively to

$$^{(1)}R_{ij} : \alpha = 1, \beta = 0, \gamma = 0, \delta = 0, \epsilon = -\tfrac{2}{3}, \varphi = \tfrac{2}{3}, \mu = -\tfrac{4}{3}, \nu = 0$$
$$^{(2)}R_{ij} : \alpha = 1, \beta = \gamma = \delta = 0, \epsilon = \tfrac{2}{3}, \varphi = \tfrac{1}{3}, \mu = -\tfrac{2}{3}, \nu = -\tfrac{1}{3}$$
$$^{(3)}R_{ij} : \alpha = 1, \beta = \tfrac{1}{2}, \gamma = \delta = 0, \epsilon = \tfrac{2}{3}, \varphi = \tfrac{1}{3}, \mu = -\tfrac{2}{3}, \nu = -\tfrac{1}{3}.$$

(c) The tensors

$$R_{ih}^{*} = R_{hi} = \Gamma_{hi,m}^{m} - \Gamma_{hm,i}^{m} + \Gamma_{lm}^{m}\Gamma_{hi}{}^{l} - \Gamma_{li}{}^{m}\Gamma_{hm}^{l}$$

$$\overline{R}_{ih} = \Gamma_{hi,m}^{m} - \Gamma_{mi,h}^{m} + \Gamma_{ml}^{m}\Gamma_{hi}{}^{l} - \Gamma_{hl}{}^{m}\Gamma_{mi}^{l}$$

$$\overline{R}_{ih}^{*} = \overline{R}_{hi} = \Gamma_{ih,m}^{m} - \Gamma_{mh,i}^{m} + \Gamma_{ml}^{m}\Gamma_{ih}{}^{l} - \Gamma_{il}{}^{m}\Gamma_{mh}^{l} \quad (15)$$

considered, for instance, by Winogradzki.[11] They correspond to

$$R_{ih}^{*} : \alpha = 1, \beta = 1, \gamma = -2, \delta = 0, \epsilon = 1, \varphi = -1, \mu = 2, \nu = 0$$

$$\overline{R}_{ih} : \alpha = 1, \beta = 0, \gamma = -2, \delta = 0, \epsilon = 2, \varphi = 0, \mu = 4, \nu = 0$$

$$\overline{R}_{ih}^{*} : \alpha = 1, \beta = 1, \gamma = 0, \delta = 0, \epsilon = 1, \varphi = 1, \mu = -2, \nu = 0.$$

In order to obtain the field equations, we consider the variational principle

$$\delta \int T_{ih}G^{ih}d\tau = 0 \quad (d\tau = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \quad (16)$$

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* See reference 10, p. 129.

where $g_{ih}$ and $\Gamma_{ih}{}^m$ are to be varied independently of one another, and their variations are chosen so that they vanish at the boundaries of integration. Following the classical method of Euler, eq. (16) gives us

$$\int\left[G^{ih}\left(\frac{\partial T_{ih}}{\partial\Delta^r_{qs}}\delta\Delta^r_{qs}+\frac{\partial T_{ih}}{\partial\Delta^r_{qs,t}}\delta\Delta^r_{qs,t}+\frac{\partial T_{ih}}{\partial S^r_{qs}}\delta S^r_{qs}+\frac{\partial T_{ih}}{\partial S^r_{qs,t}}\delta S^r_{qs,t}\right)+T_{ih}\delta G^{ih}\right]d\tau=0. \tag{17}$$

By partial integration of the second and fourth addend, and taking into account that the variation of the connection on the boundaries is zero, we get

$$\int(M_r{}^{qs}\delta\Delta^r_{qs}+N_r{}^{qs}\delta S^r_{qs}+T_{qs}\delta G^{qs})d\tau=0, \tag{18}$$

where

$$M_r{}^{qs}=G^{ih}\left(\frac{\partial T_{ih}}{\partial\Delta^r_{qs}}-\left(\frac{\partial T_{ih}}{\partial\Delta^r_{qs,t}}\right)_{,t}\right),$$
$$N_r{}^{qs}=G^{ih}\left(\frac{\partial T_{ih}}{\partial S^r_{qs}}-\left(\frac{\partial T_{ih}}{\partial S^r_{qs,t}}\right)_{,t}\right). \tag{19}$$

The skewsymmetric part of $M_r{}^{qs}$ and the symmetric part of $N_r{}^{qs}$ vanish because of the symmetry of $\Delta^r_{qs}$ and the skewsymmetry of $S^r_{qs}$ with respect to the dummy indices q, s. Consequently, we deduce from eq. (18)

$$M_r{}^{(qs)}=0,\qquad N_r{}^{[qs]}=0,\qquad T_{qs}=0, \tag{20}$$

and putting

$$M_r{}^{(qs)}+N_r{}^{[qs]}=K_r{}^{qs}, \tag{21}$$

we find that the field equations deduced from the variational principle (16) are

$$K_r{}^{qs}=0,\qquad T_{qs}=0. \tag{22}$$

By a rather long, but straightforward computation, eqs. (12) and (19) give

$$\begin{aligned}K_r{}^{qs}\equiv{}&\alpha(-G_{|r}{}^{qs}+G^{qs}S_r+\delta^s_r(G_{|t}{}^{qt}+G^{qi}S_i))-\beta(F_{,t}{}^{qt}\delta^s_r\\&+F_{,t}{}^{st}\delta_r{}^q)+\gamma(-F_{|r}{}^{qs}+F^{ih}S_{ih}{}^q\delta^s_r+F^{qs}S_r)+\delta(H^{qi}S_{ir}^{\ s}\\&-H^{si}S_{ir}{}^q)+\epsilon(-\tfrac{1}{2}(G_{|t}{}^{qt}+G^{qi}S_i)\delta^s_r+\tfrac{1}{2}(G_{|t}{}^{st}+G^{si}S_i)\delta_r{}^q\\&-G^{qs}S_r)+\varphi(\tfrac{1}{2}(-G_{,t}{}^{tq}-G^{ih}\Gamma_{hi}{}^q)\delta^s_r+\tfrac{1}{2}(G_{,t}{}^{ts}\\&+G^{ih}\Gamma_{hi}{}^s)\delta_r{}^q-G^{sq}S_r)+\mu(\tfrac{1}{2}F^{ih}S_{ih}{}^q\delta^s_r-\tfrac{1}{2}F^{ih}S_{ih}{}^s\delta_r{}^q\\&+F^{qs}S_r)+\nu(H^{qi}S_i\delta^s_r-H^{si}S_i\delta_r{}^q).\end{aligned} \tag{23}$$

A first question is to ask for the necessary and sufficient conditions in order that the system (22) be satisfied for any set of arbitrary constants $\alpha, \beta, \gamma, \ldots, \nu$. In a previous paper,[8] we proved that these conditions are

$$G_{|r}{}^{qs} = 0, \quad S_{ir}{}^{q}G^{si} - S_{ir}^{s}G^{iq} = 0,$$
$$S_{ir}{}^{q}S_{hq}^{r} = 0, \quad R_{ih} = 0, \tag{24}$$

which form a clearly incompatible system. It is, therfore, necessary to choose some particular constants $\alpha, \beta, \gamma, \ldots, \nu$. The choice will be made taking into account the simplifications they cause in the expression shown in eq. (23).

## 4. A Condensed Form of the Field Equations

We deduce from eqs. (22) and (23)

$$\begin{aligned} K_i{}^{is} = {} & (2\alpha - 5\beta + \gamma + {}^3\!/_2\, \epsilon - {}^3\!/_2\varphi)F_{,i}{}^{si} + {}^3\!/_2(\epsilon + \varphi)H_{,i}{}^{si} \\ & + (\epsilon - \varphi - \mu)S_i F^{si} + {}^3\!/_2(\epsilon - \varphi - \mu)S_{ih}{}^{s}F^{ih} \\ & - (\delta + \epsilon + \varphi + 3\nu)S_i H^{si} + {}^3\!/_2(\epsilon + \varphi)\Delta_{ih}{}^{s}H^{ih} = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} K_i{}^{qi} = {} & (3\alpha - 5\beta - \gamma - {}^3\!/_2\epsilon + {}^3\!/_2\varphi)F_{,i}{}^{qi} + {}^3\!/_2(2\alpha - \epsilon - \varphi)H_{,i}{}^{qi} \\ & + (2\alpha + 2\gamma - \epsilon + \varphi + \mu)S_i F^{qi} + {}^3\!/_2(2\alpha + 2\gamma - \epsilon + \varphi \\ & + \mu)S_{ih}{}^{q}F^{ih} + (2\alpha + \delta - \epsilon - \varphi + 3\nu)S_i H^{qi} \\ & + {}^3\!/_2(2\alpha - \epsilon - \varphi)\Delta_{ih}{}^{q}H^{ih} = 0. \end{aligned} \tag{26}$$

If we find the value of $H_{,i}{}^{si}$ from eq. (25), the value of $H_{,i}{}^{qi}$ from eq. (26), and substitute them in eq. (23), the resulting equation may be written (assuming $\alpha \neq 0, \alpha + \gamma \neq 0$),

$$\begin{aligned} Q_r{}^{qs} = {} & -(\alpha + \gamma)F_{|r}{}^{qs}(^*L) - \alpha H_{|r}{}^{qs}(^{**}L) + {}^1\!/_3(\gamma + 2\beta)\delta_r^s F_{,i}{}^{qi} \\ & + {}^1\!/_3(-2\alpha + 2\beta - \gamma)\delta_r{}^{q}F_{,i}{}^{si} = 0 \end{aligned} \tag{27}$$

where the mixed covariant derivatives refer to the connections

$$^*L_{ir}{}^{q} = \Gamma_{ir}{}^{q} + \frac{1}{3}\left(2 - \frac{\epsilon - \varphi - \mu}{\alpha + \gamma}\right)\delta_i{}^{q}S_r - \frac{1}{3}\,\frac{\epsilon - \varphi - \mu}{\alpha + \gamma}\,\delta_r{}^{q}S_i \tag{28}$$

$$\begin{aligned} ^{**}L_{ir}{}^{q} = {} & \Delta_{ir}{}^{q} + \left(1 + \frac{\delta}{\alpha}\right)S_{ir}{}^{q} + \frac{1}{3}\left(2 + \frac{\delta - \epsilon - \varphi}{\alpha}\right)\delta_i{}^{q}S_r \\ & - \frac{1}{3}\,\frac{\delta + \epsilon + \varphi}{\alpha}\,\delta_r{}^{q}S_i \end{aligned} \tag{29}$$

which satisfy the conditions

$$^*L_i = {}^*L_{[iq]}{}^{q} = 0, \qquad ^{**}L_i = {}^{**}L_{[iq]}{}^{q} = 0. \tag{30}$$

By virtue of eqs. (25) and (26), we have $K_i{}^{iq} + K_i{}^{qi} = -3Q_i{}^{qi}$, and, therefore, the field equations may be written

$$Q_r{}^{qs} = 0, \quad K_i{}^{is} = 0, \quad T_{ih} = 0, \tag{31}$$

where $Q_r{}^{qs}$, $K_i{}^{is}$, and $T_{ih}$ are given by eqs. (27), (25), and (12), respectively.

Note that equations (25) may be written

$$\begin{aligned} K_i{}^{is} &= {}^3\!/_2(\epsilon - \varphi - \mu)F_{|i}{}^{si}({}^*L) + (2\alpha - 5\beta + \gamma + {}^3\!/_2\mu)F_{,i}{}^{si} \\ &+ {}^3\!/_2(\epsilon + \varphi)H_{|i}{}^{si}({}^{**}L) \\ &- \left[\frac{\epsilon + \varphi}{\alpha}(2\alpha - \epsilon - \varphi) + \delta + 3\nu\right]S_iH^{si} = 0, \end{aligned} \tag{32}$$

and that the sum $Q_i{}^{is} + Q_i{}^{si} = 0$ and the difference $Q_i{}^{is} - Q_i{}^{si} = 0$ give

$$\alpha H_{|i}{}^{si}({}^{**}L) = {}^5\!/_3(2\beta - \alpha)F_{,i}{}^{si}, \quad F_{|i}{}^{si}({}^*L) = F_{,i}{}^{si}.$$

If we substitute in eq. (32), we get

$$K_i{}^{is} = AF_{,i}{}^{si} + BS_iH^{si} = 0, \tag{33}$$

where

$$\begin{aligned} A &= 2\alpha - 5\beta + 5\beta\frac{\epsilon + \varphi}{\alpha} - \epsilon - 4\varphi + \gamma, \\ B &= -\frac{\epsilon + \varphi}{\alpha}(2\alpha - \epsilon - \varphi) - \delta - 3\nu, \end{aligned} \tag{34}$$

which is a simple form for the second set of equations of the system (31).

## 5. Some Possible Simplifications of the Field Equations

(a) From the mathematical point of view, a first simplification is obtained if we assume the condition ${}^*L_{ir}{}^q = {}^{**}L_{ir}{}^q$. The conditions for this are

$$\delta = 0, \quad (2\varphi + \mu)\alpha + (\epsilon + \varphi)\gamma = 0 \tag{35}$$

and the unique connection $L_{ir}{}^q = {}^*L_{ir}{}^q = {}^{**}L_{ir}{}^q$, then takes the form

$$L_{ir}{}^q = \Gamma_{ir}{}^q + \frac{1}{3}\left(2 - \frac{\epsilon + \varphi}{\alpha}\right)\delta_i{}^qS_r - \frac{1}{3}\frac{\epsilon + \varphi}{\alpha}\delta_r{}^qS_i. \tag{36}$$

In order to refer the field equations to the unique connection $L_{ir}{}^q$, we must introduce it in the expression of $T_{ih}$. By direct computation, and taking (35) into account, we obtain

$$T_{ih} = T_{ih}(L) + {}^1\!/_3A(S_{i,h} - S_{h,i}) - {}^1\!/_3BS_iS_h, \tag{37}$$

where $T_{ih}(L)$ indicates the tensor (12) written with respect to the connection (36), and A, B are given by (34).

In order to eliminate the vector $S_i$ we must add the further condition $B = 0$, i.e.,

$$\frac{\epsilon + \varphi}{\alpha}(2\alpha - \epsilon - \varphi) + \delta + 3\nu = 0. \tag{38}$$

If this condition is satisfied, the third set of equations (31) may be replaced by

$$T_{(ih)} = 0, \quad T_{[ih],j} + T_{[hj],i} + T_{[ji],h} = 0. \tag{39}$$

With the conditions of (35) and (38), the equations (33) give $F_{,i}{}^{si} = 0$, and the equations $Q_r{}^{qs} = 0$ (since they imply the vanishing of the symmetric and skewsymmetric parts of the lefthand side), give $F|_r{}^{qs}(L) = 0$, $H|_r{}^{qs}(L) = 0$, i.e., $G|_r{}^{qs} = 0$. Therefore, the field equations take the form

$$\begin{aligned} &G|_r{}^{qs} = 0, \quad F_{,i}{}^{si} = 0 \\ &T_{(ih)} = 0, \quad T_{[ih],j} + T_{[hj],i} + T_{[ji],h} = 0, \end{aligned} \tag{40}$$

which is the form of the weak system of Einstein (7) with the Ricci tensor $R_{ih}$ substituted for by the general tensor $T_{ih}$. The mixed covariant derivative of the first set of equations in (40) refers to the connection shown in (36). We have proved the following:

THEOREM 2: *In order that the field equations deduced from the variational principle* (16) *reduce to the Einstein system* (40), *it is necessary and sufficient that the coefficients of the tensor* $T_{ih}$ (12) *satisfy the conditions* (35) *and* (38).

(b) *Invariance by λ-transformations.* The conditions (35) and (38) have been introduced in order to simplify the field equations. It is interesting to see if they may be deduced from geometrical hypothesis.

It is well known that the most general change of connection which preserves parallelism is

$$\Gamma_{ih}{}^m \to \Gamma_{ih}{}^m + \delta_i{}^m \lambda_h, \tag{41}$$

where $\lambda_h$ is an arbitrary covariant vector.* Einstein proposes to require the field equations to exhibit the property of λ-invariance, that is, the hypothesis that they should be invariant by transformations of the type shown in eq. (41), called λ-transformations.[3]

Through the use of the λ-transformation (41), the vector $S_i$ and the connections (28) and (29) transform in accordance with the following formulae:

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* See reference 4, p. 30.

$$S_i \to S_i - \tfrac{3}{2}\lambda_i$$

$$*L_{ih}{}^m \to *L_{ih}{}^m + \frac{1}{2}\,\frac{\epsilon - \varphi - \mu}{\alpha + \gamma}\,(\delta_i{}^m\lambda_h + \delta_h{}^m\lambda_i)$$

$$**L_{ih}{}^m \to **L_{ih}{}^m + \frac{1}{2}\,\frac{\epsilon + \varphi}{\alpha}(\delta_i{}^m\lambda_h + \delta_h{}^m\lambda_i). \qquad (42)$$

It may then be ascertained by direct computation that $T_{ih}$ transforms in accordance with

$$\begin{aligned} T_{ih} \to T_{ih} &+ \tfrac{1}{2}(2\alpha - 5\beta + \gamma - 3\varphi)\lambda_{h,i} - \tfrac{1}{2}(2\alpha - 5\beta + \gamma + 3\epsilon)\lambda_{i,h} \\ &+ \tfrac{1}{2}(-\delta - 2\epsilon + \mu - 3\nu)S_i\lambda_h \\ &+ \tfrac{1}{2}(-\delta - 2\varphi - \mu - 3\nu)S_h\lambda_i \\ &+ \tfrac{3}{4}(\delta + 2\epsilon + 2\varphi + 3\nu)\lambda_i\lambda_h \\ &+ \tfrac{3}{2}(\epsilon - \varphi - \mu)S_{ih}{}^m\lambda_m \\ &+ \tfrac{3}{2}(\epsilon + \varphi)\Delta_{ih}{}^m\lambda_m. \end{aligned} \qquad (43)$$

From eqs. (42) and (43) we deduce that the conditions for the $\lambda$-invariance of the system shown in (31) are

$$\epsilon + \varphi = 0, \quad \epsilon - \varphi - \mu = 0, \quad \delta + 3\nu = 0 \qquad (44)$$

$$2\alpha - 5\beta + \gamma - 3\varphi = 0. \qquad (45)$$

In this case, equations (33) are identically satisfied, and the system (31) reduces to $Q_r{}^{qs} = 0, T_{ih} = 0$. If we accept that $T_{ih}$ and its transform via a $\lambda$-transformation differ by a curl, the conditions above reduce to those shown in (44), and the field equations become

$$F|_r{}^{qs}(*L) = 0, \quad H|_r{}^{qs}(**L) = 0, \quad F_{,i}{}^{si} = 0$$

$$T_{(ih)} = 0, \; T_{[ih],j} + T_{[hj],i} + T_{[ji],h} = 0. \qquad (46)$$

The only extra condition necessary for this system to coincide with the Einstein system is $\delta = 0$. We therefore have the following:

**THEOREM 3:** *The condition of $\lambda$-invariance to within a curl for $T_{ih}$, plus the condition $\delta = 0$, reduces the field equations of the variational principle* (16) *to the Einstein system* (40), *the tensor $T_{ih}$ being any one of the form* (12) *with the conditions of* (44) *and $\delta = 0$.*

The conditions above are more restrictive than those of theorem 2; that is, there are tensors $T_{ih}$ which lead to the Einstein system without possessing the property of $\lambda$-invariance. That is the case, for instance, with the tensor ${}^{(3)}R_{ih}$ of (14).

(c) *Conditions of Pseudo-Hermitian Symmetry.* A tensor of rank 2 which depends on $g_{ij}$ and $\Gamma_{ih}{}^{m}$ is called pseudo-Hermitian if it remains unchanged when we replace $g_{ij}$ by $g_{ji}$, and $\Gamma_{ih}{}^{m}$ by $\Gamma_{hi}{}^{m}$.[2,*] Suppose that we impose upon the tensor $T_{ih}$ the condition of being pseudo-Hermitian. It can be obtained, by a direct computation, that the conditions for this are the following:

$$\alpha = 2\beta, \quad \alpha = \varphi + \epsilon, \quad \gamma + \mu + 2\varphi = 0. \tag{47}$$

Among the pseudo-Hermitian tensors we have the Einstein tensor (13) and the tensor ${}^{(3)}R_{ij}$ (14). This last tensor also satisfies the conditions of theorem 2, i.e., it gives rise to a system of field equations of the Einstein type. The conditions for a tensor $T_{ih}$ of the form (12) to be pseudo-Hermitian and lead to the system (40) of Einstein, are (47), (38) and (35).

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* See reference 5, p. 54.