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Source: *The American Mathematical Monthly*, Vol. 56, No. 4 (Apr., 1949), pp. 270-271

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2304779>

Accessed: 29/09/2008 06:05

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The coördinates of A_i, B_i, C_i ($i=1, 2$) are easily found to be (r, s, t) where r, s, t are distinct permutations of $0, \lambda, \mu$. Evidently all six points lie on the ellipse whose areal equation is

$$S_1 \equiv \lambda\mu(x^2 + y^2 + z^2) - (\lambda^2 + \mu^2)(yz + zx + xy) = 0.$$

Moreover we have the equation of the Steiner ellipse*

$$S_2 \equiv yz + zx + xy = 0.$$

Thus

$$S_1 = -(\lambda + \mu)^2 S_2 + \lambda\mu u^2 - (\lambda + \mu)^2 (S_2 - ku^2),$$

where $u = x + y + z$. Hence S_1, S_2 are homothetic and concentric with the centroid G as their homothetic center.† To find the ratio of homothety, H , let S_2 divide the line joining G to A in ratio $m:n$. We have

$$\frac{m}{n} = \frac{(\lambda^2 - \lambda\mu + \mu^2) \pm (\lambda + \mu)\sqrt{\lambda^2 - \lambda\mu + \mu^2}}{3\lambda\mu}$$

and

$$H = \frac{m}{m+n} = \frac{\sqrt{\lambda^2 - \lambda\mu + \mu^2}}{\lambda + \mu}.$$

When $\lambda:\mu = 1:2$, then $H = 1/\sqrt{3}$.

We note that as the ratio $\lambda:\mu$ varies, the equation S_1 represents a system of homothetic and concentric ellipses with the centroid G as their homothetic center. Since the ratio $\lambda:\mu$ and $\mu:\lambda$ are simultaneously treated in our solution, we need only consider the ratio $\lambda:\mu$ in the interval $(-1, 1)$. Limiting cases arise (i) when $\lambda:\mu = 1$, giving the maximum inscribed ellipse;‡ (ii) when $\lambda = 0$, giving the minimum circumscribed ellipse (Steiner ellipse); and (iii) when $\lambda:\mu = -1$, giving the line at infinity. It is also of interest that for real values of λ and μ , the system of ellipses fills the plane except for the interior of the maximum inscribed ellipse.

Also solved by L. M. Kelly and R. Goormaghtigh.

Rectifiable Plane Curves

4262 [1947, 418]. Proposed by L. A. Santaló, Rosario, Argentina

Let C be a rectifiable plane curve of length L , contained within a given circle of radius R . Prove that there is a circle of radius $\rho \geq R$ which cuts C in n points, where

$$(1) \quad n \geq L/\pi R.$$

* D. M. Y. Sommerville, *Analytical Conics*, London, 1924, p. 191.

† D. M. Y. Sommerville, loc. cit., p. 205.

‡ D. M. Y. Sommerville, loc. cit., p. 181.

In particular there is a line which cuts C in n points, where n satisfies (1). If $\rho < R$, the inequality (1) must be replaced by

$$(2) \quad n \geq \frac{4L\rho}{\pi(R + \rho)^2}.$$

See L. A. Santaló, A theorem and an inequality referring to rectifiable curves, *American Journal of Mathematics*, 1941, p. 635.

Solution by the Proposer. Let (x, y) be the coördinates of the variable center of a circle of constant radius ρ . Let $N \equiv N(x, y)$ be the number of common points of this circle and the curve C for each position of (x, y) . Then the Proposer has shown, in the paper already cited, that the following integral formula holds;

$$\iint N dx dy = 4L\rho.$$

On the other hand, if $\rho \geq R$, the area covered by the points (x, y) which are centers of circles of radius ρ and which cut the given circle of radius R , has the value

$$\pi(\rho + R)^2 - \pi(\rho - R)^2 = 4\pi R\rho.$$

Consequently the mean value of N is

$$\bar{N} = \frac{\iint N dx dy}{\iint dx dy} = \frac{L}{\pi R}.$$

As the mean value of a function is not greater than its maximum value, the inequality (1) is established.

If $\rho < R$, the area covered by the centers (x, y) of circles of radius ρ which cut the given circle of radius R or are contained in its interior is $\pi(\rho + R)^2$. Consequently $\bar{N} = 4L\rho/\pi(\rho + R)^2$ and inequality (2) holds.

The Continuum Hypothesis

4263 [1947, 419]. Proposed by Howard Eves, Oregon State College, and Paul Halmos, Syracuse University

Criticize the following alleged proof of the continuum hypothesis.

Let X be the set of all infinite sequences of 0's and 1's, and let E be an arbitrary uncountable subset of X . Corresponding to any finite sequence, $\{a_1, \dots, a_k\}$, of 0's and 1's, write $E(a_1, \dots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to E and begin with $\{a_1, \dots, a_k\}$. Since $E = E(0) + E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1 = 0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If a_i has already been defined for $i = 1, \dots, k$, so that $E(a_1, \dots, a_k)$ is uncountable, then write $a_{k+1} = 0$ or 1 according as $E(a_1, \dots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \dots\}$ has the property