The coördinates of A_i, B_i, C_i $(i=1, 2)$ are easily found to be (r, s, t) where r, s, t are distinct permutations of $0, \lambda, \mu$. Evidently all six points lie on the ellipse whose areal equation is

$$S_1 \equiv \lambda\mu(x^2 + y^2 + z^2) - (\lambda^2 + \mu^2)(yz + zx + xy) = 0.$$

Moreover we have the equation of the Steiner ellipse*

$$S_2 \equiv yz + zx + xy = 0.$$

Thus

$$S_1 = -(\lambda + \mu)^2 S_2 + \lambda\mu u^2 - (\lambda + \mu)^2(S_2 - ku^2),$$

where $u = x+y+z$. Hence S_1, S_2 are homothetic and concentric with the centroid G as their homothetic center.† To find the ratio of homothety, H, let S_2 divide the line joining G to A in ratio $m:n$. We have

$$\frac{m}{n} = \frac{(\lambda^2 - \lambda\mu + \mu^2) \pm (\lambda + \mu)\sqrt{\lambda^2 - \lambda\mu + \mu^2}}{3\lambda\mu}$$

and

$$H = \frac{m}{m+n} = \frac{\sqrt{\lambda^2 - \lambda\mu + \mu^2}}{\lambda + \mu}.$$

When $\lambda:\mu = 1:2$, then $H = 1/\sqrt{3}$.

We note that as the ratio $\lambda:\mu$ varies, the equation S_1 represents a system of homothetic and concentric ellipses with the centroid G as their homothetic center. Since the ratio $\lambda:\mu$ and $\mu:\lambda$ are simultaneously treated in our solution, we need only consider the ratio $\lambda:\mu$ in the interval $(-1, 1)$. Limiting cases arise (i) when $\lambda:\mu = 1$, giving the maximum inscribed ellipse;‡ (ii) when $\lambda = 0$, giving the minimum circumscribed ellipse (Steiner ellipse); and (iii) when $\lambda:\mu = -1$, giving the line at infinity. It is also of interest that for real values of λ and μ, the system of ellipses fills the plane except for the interior of the maximum inscribed ellipse.

Also solved by L. M. Kelly and R. Goormaghtigh.

Rectifiable Plane Curves

4262 [1947, 418]. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C be a rectifiable plane curve of length L, contained within a given circle of radius R. Prove that there is a circle of radius $\rho \geqq R$ which cuts C in n points. where

(1) $$n \geqq L/\pi R.$$

* D. M. Y. Sommerville, *Analytical Conics*, London, 1924, p. 191.

† D. M. Y. Sommerville, loc. cit., p. 205.

‡ D. M. Y. Sommerville, loc. cit., p. 181.

In particular there is a line which cuts C in n points, where n satisfies (1). If $\rho < R$, the inequality (1) must be replaced by

$$n \geqq \frac{4L\rho}{\pi(R+\rho)^2}. \tag{2}$$

See L. A. Santaló, A theorem and an inequality referring to rectifiable curves, *American Journal of Mathematics*, 1941, p. 635.

Solution by the Proposer. Let (x, y) be the coördinates of the variable center of a circle of constant radius ρ. Let $N \equiv N(x, y)$ be the number of common points of this circle and the curve C for each position of (x, y). Then the Proposer has shown, in the paper already cited, that the following integral formula holds;

$$\iint N dxdy = 4L\rho.$$

On the other hand, if $\rho \geqq R$, the area covered by the points (x, y) which are centers of circles of radius ρ and which cut the given circle of radius R, has the value

$$\pi(\rho + R)^2 - \pi(\rho - R)^2 = 4\pi R\rho.$$

Consequently the mean value of N is

$$\overline{N} = \frac{\iint N dxdy}{\iint dxdy} = \frac{L}{\pi R}.$$

As the mean value of a function is not greater than its maximum value, the inequality (1) is established.

If $\rho < R$, the area covered by the centers (x, y) of circles of radius ρ which cut the given circle of radius R or are contained in its interior is $\pi(\rho+R)^2$. Consequently $\overline{N} = 4L\rho/\pi(\rho+R)^2$ and inequality (2) holds.

The Continuum Hypothesis

4263 [1947, 419]. *Proposed by Howard Eves, Oregon State College, and Paul Halmos, Syracuse University*

Criticize the following alleged proof of the continuum hypothesis.

Let X be the set of all infinite sequences of 0's and 1's, and let E be an arbitrary uncountable subset of X. Corresponding to any finite sequence, $\{a_1, \cdots, a_k\}$, of 0's and 1's, write $E(a_1, \cdots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to E and begin with $\{a_1, \cdots, a_k\}$. Since $E = E(0) + E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1 = 0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If a_i has already been defined for $i = 1, \cdots, k$, so that $E(a_1, \cdots, a_k)$ is uncountable, then write $a_{k+1} = 0$ or 1 according as $E(a_1, \cdots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \cdots\}$ has the property