ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4259. *Proposed by Richard Bellman, Princeton University*

If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1 + x^{n_k}} = x \prod_{k=1}^{\infty} (1 + x^{n_k}), \qquad |x| < 1,$$

show that, except perhaps for order,

$$n_k = 2^k.$$

4260. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC inscribe two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose sides are parallel to the medians. Show that (1) the triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$, have the same centroid and the same Brocard angle; (2) the triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in an ellipse concentric and homothetic to the inscribed Steiner ellipse, the ratio of homothety being $1/\sqrt{3}$.

4261. *Proposed by F. J. Dyson, Trinity College, Cambridge, England*

The number of partitions of an integer n into a sum of positive integral parts is denoted by $p(n)$. The result of subtracting the number of parts in a partition from the largest part is a positive or negative integer called the rank of the partition. Ramanujan proved that $p(5n+4)$ is always divisible by 5, and $p(7n+5)$ by 7. Show that the number of partitions of $5n+4$ whose ranks are congruent modulo 5 to a given residue is the same whichever of the five residues is chosen, and the number of partitions of $7n+5$ whose ranks are congruent modulo 7 to a given residue is the same whichever of the seven residues is chosen.

4262. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C be a rectifiable plane curve of length L, contained within a given circle of radius R. Prove that there is a circle of radius $\rho \geqq R$ which cuts C in n points, where

$$(1) \qquad n \geqq L/\pi R.$$

In particular there is a line which cuts C in n points, where n satisfies (1). If

$\rho < R$, the inequality (1) must be replaced by

$$(2) \qquad n \geqq \frac{4L\rho}{\pi(R+\rho)^2}.$$

See L. A. Santaló, A theorem and an inequality referring to rectifiable curves, *American Journal of Mathematics*, 1941, p. 635.

4263. *Proposed by Howard Eves, Oregon State College, and Paul Halmos, Syracuse University*

Criticize the following alleged proof of the continuum hypothesis.

Let X be the set of all infinite sequences of 0's and 1's, and let E be an arbitrary uncountable subset of X. Corresponding to any finite sequence, $\{a_1, \cdots, a_k\}$, of 0's and 1's, write $E(a_1, \cdots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to E and begin with $\{a_1, \cdots, a_k\}$. Since $E=E(0)+E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1=0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If a_i has already been defined for $i=1, \cdots, k$, so that $E(a_1, \cdots, a_k)$ is uncountable, then write $a_{k+1}=0$, or 1 according as $E(a_1, \cdots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \cdots\}$ has the property that for any k it is true that $E(a_1, \cdots, a_k)$ is uncountable. Write E^* for the union of all $E(a_1, \cdots, a_k)$, for $k=1, 2, 3, \cdots$; then E^* is a subset (in fact an uncountable subset) of E.

For certain positive integers k it is true that both $E(a_1, \cdots, a_k, 0)$ and $E(a_1, \cdots, a_k, 1)$ are uncountable; in fact this must happen for an infinite number of k's. (Otherwise, for a sufficiently large k, $E(a_1, \cdots, a_k)$ would not be uncountable, contrary to its construction.) Let $k_1, k_2, k_3, \cdots$ be the integers for which this is true, and write, for any $\{x_1, x_2, x_3, \cdots\}$ in E^*,

$$y_n = x_{k_n+1};$$

then $\{y_1, y_2, \cdots\}$ is an infinite sequence of 0's and 1's. From the way in which the k_n are defined it follows that every possible sequence of 0's and 1's occurs as a y sequence, and that consequently the sequences $\{x_1, x_2, \cdots\}$ in E^* correspond (in possibly a many to one manner) to a set (viz. the set of all y sequences) having the power of the continuum. It follows that the cardinal number of E^* (and hence of E) cannot be less, and since E is a subset of X it cannot be greater. In other words it has been proved that every uncountable subset of a set having the power of the continuum has also the power of the continuum.

4248 [1947, 232], corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given a tetrahedron $ABCD$, place a sphere (S) of given radius in such a manner that the volume of the polar tetrahedron of $ABCD$ with respect to (S) will be a relative minimum.