that is, an nth order determinant Δ in which $(2k+1)$ consecutive diagonals symmetrical about and including the principal diagonal have *no* zero elements, while all other elements are zero in the two equal triangular corners of the determinant at each side of the central diagonal band. How many non-zero terms are there in the completely expanded determinant?

4208. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron, if two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere whose center is the complement of the orthocenter for the linear net determined by the spheres circumscribed and conjugate.

SOLUTIONS

De L'Hospital's Rule

4151 [1945, 163]. *Proposed by B. M. Stewart, Michigan State College*

Let O be a point at which a given curve has a second derivative; let the tangent and normal at O serve respectively as x, y axes, the equation of the curve becoming $y=f(x)$; and let P; (x, y) be a point on the curve, say with positive x. Denote the arc length OP by s and locate the point S: $(s, 0)$. The line SP intersects the y-axis in the point B:$(0, b)$. If R indicates $[1+(y')^2]^{3/2}/y''$, show that the limiting position of the point B as P approaches O is such that lim $b=3$ lim R.

This is a generalization of a problem in the calculus of Granville, Smith, and Longley, 1934, p. 177, ex. 20.

Solution by L. A. Santaló, Rosario, Argentina. Let $x=x(s)$, $y=y(s)$ be the parametric equations of the curve $y=f(x)$ with the arc length s as parameter. If we denote by accents derivatives with respect to s, it is well known that $(x')^2+(y')^2=1$ and $x'x''+y'y''=0$. From this and from $y'((0)=0$ and $R=1/y''$, we deduce $x''(0)=0$. Hence

$$(1)\qquad x(0) = y(0) = 0;\ \ x'(0) = 1,\ y'(0) = 0;\ \ x''(0) = 0,\ y''(0) = 1/R_0;$$

where

$$R_0 = \lim_{s\to 0} R.$$

From the equation of the straight line SP we deduce $b=OB=sy/(s-x)$. Applying l'Hospital's rule twice, we have

$$\lim_{s\to 0} b = \lim_{s\to 0}\frac{sy}{s-x} = \lim_{s\to 0}\frac{y+sy'}{1-x'} = \lim_{s\to 0}\frac{2y'+sy''}{-x''}.$$

From the fact that $x'x''=-y'y''$, and from (1), we deduce that

$$\lim_{s\to 0} b = \lim_{s\to 0}\frac{2y'x'+sy''x'}{y'y''} = 2R_0 + \lim_{s\to 0}\frac{sx'}{y'},$$

and, if we apply this rule again, we find that

$$\lim_{s\to 0} b = 3R_0.$$

This problem may be generalized in the following way. Let us consider two plane curves C, C_1 tangent at O, and let R and R_1 be their radii of curvature at O. Let P be a point on C and P_1 a point on C_1 with the condition that arc length $OP=s$ equals arc length OP_1. The straight line of PP_1 meets the normal at O in point B; then we have

$$\lim_{s\to 0} OB = \frac{3}{\dfrac{1}{R}+\dfrac{1}{R_1}}\cdot \tag{2}$$

If $R_1=\infty$ we obtain the proposed problem. If $R=R_1$, the formula (2) is not applicable. In this case, if $(dR/ds)_0 \neq (dR_i/ds)_0$, we find that

$$\lim_{s\to 0} OB = \tfrac{4}{3}R. \tag{3}$$

From (2) and (3) may be deduced a geometrical example referring to a function of two variables for which the order of limits cannot be interchanged. Let us consider the curves C and C_1 tangent at O with $R\neq R_1$, and suppose that we wish the limiting position of the point B as $R_1\to R$ and $s\to 0$. If we take first $s\to 0$ and then $R_1\to R$, by (2) it follows that $\lim OB=3R/2$. But if we take first $R_1\to R$ and then $s\to 0$, we have by (3) $\lim OB=4R/3$.

All these results and the proposed problem are treated in our paper *Algunas propriedades infinitesimales de las curvas planas*, Mathematicae Notae, Año I, pp. 128–144, 1941.

Solved also by Mrs. R. C. Buck, Howard Eves, J. F. Heyda, J. B. Kelly, A. Sisk, C. E. Springer, J. T. Webster, and the proposer.

Editorial Note. The solutions by Eves and Springer used formulas for the coordinates x, y of a point on a curve in terms of the arc length parameter s, the first referring to Graustein's *Differential Geometry*, p. 39, equations (46), and the second to Eisenhart's *An Introduction to Differential Geometry*, p. 26. The remaining solvers use three applications of de l'Hospital's theorem in a manner somewhat similar to the above solution.

We may also use vector methods as follows. Consider the curve as concave upward at O passing through P in its neighborhood, say to the right; and let $\mathbf{t}$ and $\mathbf{n}$ be the unit vector tangent and normal at O forming a right hand system. Let $\mathbf{r}$ be the vector of P, the origin of vectors being arbitrarily chosen, where $\mathbf{r}$ is a function of s, the arc length to P with the positive sense $\overset{\frown}{OP}$. Then at O we find the values

$$\begin{gathered}\mathbf{t} = d\mathbf{r}/ds = \mathbf{r}'; \ \mathbf{t}' = \mathbf{r}'' = \kappa\mathbf{n},\ \kappa = 1/R;\ \mathbf{n}' = -\kappa\mathbf{t};\ \mathbf{t}'' = \kappa'\mathbf{n} - \kappa^2\mathbf{t};\\ \mathbf{t}''' = (\kappa'' - \kappa^3)\mathbf{n} - 3\kappa\kappa'\mathbf{t};\quad etc.\end{gathered} \tag{1}$$

Using these results we have

$$\overrightarrow{OP} = d\mathbf{r} = \mathbf{t}ds + \frac{\kappa\mathbf{n}}{2!}(ds)^2 + \frac{(\kappa'\mathbf{n} - \kappa^2\mathbf{t})}{3!}(ds)^3 + \cdots. \tag{2}$$

Since $\overrightarrow{OS} = \mathbf{t}ds$, the sum of the remaining terms of (2) is $\overrightarrow{SP}$; and then $b\mathbf{n} = \mathbf{t}ds + \overrightarrow{ASP}$. This last equation gives

$$\begin{aligned} 1 &= \frac{A(ds)^2}{3!}\left[\kappa^2 + \tfrac{3}{4}\kappa\kappa' ds + \cdots\right]; \\ b &= \frac{A(ds)^2}{2}\left[\kappa + \frac{\kappa' ds}{3} + \cdots\right]; \\ b &= 3R\frac{\left[1 + \frac{R\kappa'}{3}ds + \cdots\right]}{\left[1 + \frac{3R\kappa'}{4}ds + \cdots\right]}. \end{aligned} \tag{3}$$

The limit value of b is now easily obtained.

Trigonometric Expansion of Binomial Coefficients

4152 [1945, 163]. *Proposed by William J. Taylor, Washington, D. C.*

Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m\pi x}{N}, \qquad -N < x < N.$$

I. *Solution by J. B. Kelly, Hampton, Va.* Set $w = e^{\pi i/N}$, then the right member in the problem becomes

$$\begin{aligned} &\frac{1}{2N}\sum_{n=1}^{N}(w + w^{-n})^N(w^{nx} + w^{-nx}), \\ &= \frac{1}{2N}\sum_{n=1}^{N}\sum_{k=0}^{N}\binom{N}{k} w^{n(N-2k)}(w^{nx} + w^{-nx}), \\ &= \frac{1}{2N}\sum_{k=0}^{N}\binom{N}{k}\sum_{n=1}^{N}\left[w^{n(N-2k+x)} + w^{n(N-2k-x)}\right]. \end{aligned} \tag{1}$$

Since $-N < x < N$, we have

$$\sum_{n=1}^{N} w^{n(N-2k+x)} = 0, \tag{2}$$

unless $N - 2k + x = 0$, in which case the sum in (2) is N. Similarly,