$$(p^h - 1)[(p^h)^{k-1} + (p^h)^{k-2} + \cdots + p^h + 1] = 2^n.$$

This is impossible because the bracket is also odd and greater than 1. Hence $r = 2^m$.

However, when $r = 2^m$, the given equation can be written as

$$(p^{2^{m-1}} - 1)(p^{2^{m-1}} + 1) = 2^n,$$

and it is easily proved that this is possible only when $m = 0$ or 1, and that for the latter situation $p = 3$.

It is also seen that the hypothesis that p is a prime is superfluous.

II. *Solution by D. W. Alling, Rochester, New York.* We have $p^r \equiv 1 (\text{mod } 2^n)$, and it is clear that p belongs to the exponent r mod 2^n. But $\phi(2^n) = 2^{n-1}$ and $r | 2^{n-1}$. Hence r is a power of 2. If $r > 1$, then r is even and the integral factorization

$$(p^{r/2} - 1)(p^{r/2} + 1) = 2^n$$

is possible. Therefore

$$p^{r/2} - 1 = 2^a, \qquad p^{r/2} + 1 = 2^b, \qquad 2^a - 2^b = -2.$$

Solving we find, uniquely, $b = 2$, $a = 1$, $n = 1$.

Also solved by D. W. Alling (another solution), Murray Barbour, D. H. Browne, R. C. Buck, A. Charnes, Roy Dubisch, Paul Erdös, N. J. Fine, J. B. Kelly, E. D. Schell, Peter Scherk (two ways), E. P. Starke, and the proposer (two ways).

The solutions of Murray Barbour, Roy Dubisch, N. J. Fine, J. B. Kelly, E. D. Schell, and one of the solutions of the proposer are essentially like solution I above. The alternate solution of D. W. Alling and the solution of D. H. Browne utilize the fact that all primes are of the form $4k \pm 1$. The proposer offered a second solution using the theory of Galois groups. R. C. Buck established the more general theorem: If $q^n + 1 = a^r$, *q prime, then* $r = 1$ *except for the special case* $2^3 + 1 = 3^2$. *If, further, a is prime, then the only solutions are the Fermat primes,* $p = 2^{2^n} + 1$. Peter Scherk and Paul Erdös solved essentially Buck's extension. Most solvers noted the uniqueness of solution when $r > 1$, and several observed that p need not be restricted to a prime.

An Integral Related to the Gamma Function

E 664 [1945, 159]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that if $x < 1$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \int_1^{\infty} t^n e^{-t} dt = \frac{e^{x-1}}{1 - x}.$$

I. *Solution by N. J. Fine, Purdue University.* The problem should be stated with the condition $|x| < 1$. Set $B_n = 1/n! \int_1^{\infty} t^n e^{-t} dt$. $B_0 = e^{-1}$ and an integration by by parts shows that $B_n = e^{-1}/n! + B_{n-1}$, so $B_n = e^{-1} \sum_{k=0}^{n} (1/k!)$. Hence, if $|x| < 1$,

$$\sum_{n=0}^{\infty} B_n x^n = e^{-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^n}{k!} = e^{-1}\sum_{k=0}^{\infty} \frac{1}{k!}\sum_{n=k}^{\infty} x^n = e^{-1}\sum_{k=0}^{\infty} \frac{x^k}{k!}\cdot\frac{1}{1-x} = \frac{e^{x-1}}{1-x}.$$

II. *Solution by Harley Flanders, University of Chicago.* The problem should be stated with the condition $|x| < 1$. Set $c_n = \int_1^\infty t^n e^{-t}dt$. Then we have

$$n! = \int_0^\infty t^n e^{-t}dt > c_n = \int_0^\infty t^n e^{-t}dt - \int_0^1 t^n e^{-t}dt > n! - \int_0^1 e^{-t}dt = n! - 1 + e^{-1}.$$

Therefore

$$\sum_{n=0}^{\infty} x^n > \sum_{n=0}^{\infty} (c_n x^n)/n! > \sum_{n=0}^{\infty} [1 - (1 - e^{-1})/n!]x^n.$$

By the "ratio test" we see that both extreme series, and hence the given series, converge in the interval $|x| < 1$. Hence the given series converges uniformly in that interval by a known theorem on power series and we may interchange the operations:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\int_1^\infty t^n e^{-t}dt = \int_1^\infty \left[\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right] e^{-t}dt = \int_1^\infty e^{(x-1)t}dt = \frac{e^{x-1}}{1-x},$$

(since $x - 1 < 0$).

Also solved by D. W. Alling, Murray Barbour, Ellen Buck, Sidney Glusman, J. E. Hanson, J. B. Kelly, E. E. Osborne, E. P. Starke, C. W. Topp, and J. T. Webster.

J. B. Kelly obtained his solution from results of E 654. E. P. Starke showed, as an incidental note to his solution, that

$$ec_n/n! < e < (ec_n + 1)/n!,$$
$$ec_1 = 2, \qquad ec_{n+1} = 1 + (n+1)ec_n.$$

Since the integers ec_n are easy to compute we then have here a novel way of computing the numerical value of e.

Circles Covering a Given Curve

E 665 [1945, 159]. *Proposed by L. A. Santalo, Rosario, Argentina*

Let C be a closed convex plane curve with continuous radius of curvature R. Let R_M be the greatest value of R. Given $\lambda \geqq R_M$, show that the area F_λ covered by the centers of circles of radius λ which contain C in their interior is given by

$$F_\lambda = F - L\lambda + \pi\lambda^2,$$

where L and F are the length and area of C.

Solution by R. A. Rosenbaum, U.S.N.R. Problem E 630 [1945, 160] can be easily generalized so as to include the present problem. The generalization of E 630 is: