PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

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The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 646. *Proposed by Orrin Frink, Jr., Xenia, Ohio*

Prove that any two conjugate planes through a secant of a sphere meet the sphere in orthogonal circles.

E 647. *Proposed by V. Thébault, San Sebastián, Spain*

Let (m_1, m_2, m_3, m_4) be the barycentric coordinates of a point G with respect to a regular tetrahedron $A_1A_2A_3A_4$ of edge a (so that G is the centroid of masses m_1, m_2, m_3, m_4 at A_1, A_2, A_3, A_4). Obtain an expression for the distance A_4G.

E 648. *Proposed by Mary L. Boas and R. P. Boas, Jr., Tufts College and Harvard University*

Show that, when $x=2\cos\pi/(n+1)$, the n-rowed determinant

$$\begin{vmatrix} x & 1 & 0 & 0\cdots0 & 0 \\ 1 & x & 1 & 0\cdots0 & 0 \\ 0 & 1 & x & 1\cdots0 & 0 \\ 0 & 0 & 1 & x\cdots0 & 0 \\ \cdot & \cdot & \cdot & \cdot\cdots\cdot & \cdot \\ 0 & 0 & 0 & 0\cdots x & 1 \\ 0 & 0 & 0 & 0\cdots1 & x \end{vmatrix}$$

has the value zero.

E 649. *Proposed by L. A. Santaló, Rosario, Argentine Republic*

A set of parallel line segments will be called "linear" if all of them can be cut by one straight line. Show by an example that an infinite set of parallel segments in one plane may have the property that every subset of three is linear while the whole set is not linear. (The segments are "open": not including their end points.)

E 650. *Proposed by Lloyd Dulmage, University of Manitoba*

If we first arrange n letters in a row, in a definite order, and then arrange below these letters

(a) a second row containing the same n letters so that no letter is repeated in any column, the number of possible arrays is ${}_2K_n$;

(b) a second row containing p of the letters of the first row together with $n-p$ other letters, so that no letter is repeated in any column, then the number of arrays is ${}_2K_{n,p}$;

(c) a second row containing a definite p of the letters of the first row (and $n-p$ empty spaces) so that exactly some q of these p letters do not appear below any of the p chosen letters (but the remaining $p-q$ letters appear below some $p-q$ of the p chosen letters), no letter being repeated in any column, then the number of arrays is ${}_2K_{n,p,q}$;

(d) two rows (second and third) each containing the same n letters, so that no letter is repeated in any column, then the number of arrays is ${}_3K_n$.

Show that the functions so defined satisfy the following relations:

$$(1) \qquad {}_2K_{n,p} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} (n-r)!,$$

$$(2) \qquad {}_2K_{n,p,q} = \binom{p}{q}\binom{n-p}{q} {}_2K_{p,p-q},$$

$$(3) \qquad {}_2K_n = \sum_{q=0}^{\min(p,n-p)} {}_2K_{n,p,q} \cdot {}_2K_{n-p,n-p-q}$$

(for each value of p from 0 to n),

$$(4) \qquad {}_3K_n = \sum_{p=0}^{n} \sum_{q=0}^{\min(p,n-p)} (-1)^p \binom{n}{p} {}_2K_{n,p,q}({}_2K_{n-p,n-p-q})^2.$$

SOLUTIONS

Arc and Area of an Epicycloid

E 600 [1943, 634]. *Proposed by J. H. Butchart, Grinnell College*

If the radii of the fixed and rolling circles are a and b respectively, the length of one arch of an epicycloid is $8(a+b)b/a$, and the area bounded by one arch and the fixed circle is

$$\pi(3a^2 + 8ab + 4b^2)b^2/a(a+2b).$$

Corresponding formulas for the hypocycloid are obtained by changing the sign of b. Prove these results synthetically.

Solution by the Proposer. The formulas will be proved for the epicycloid $APQR$ generated by rolling a circle (O', b) on a circle (O, a). Modifications for the hypocycloid are obvious. Consider a circle $(O'', ab/(a+2b))$ tangent to (O') and homothetic to it with respect to the center O, and let it roll on an inner circle centered at O. As the point P, fixed on (O'), describes the epicycloid, the point P', fixed on (O''), describes the evolute. This evolute, an epicycloid whose arch subtends the angle $2\pi b/a$, is clearly similar to the original epicycloid, the