\[
\sum_{j=i+1}^{i+k} \lambda_j > m(v_1 + v_2 + \cdots + v_r - l + v_{r+1} + \cdots + v_i) + 2 = mk + 2.
\]

If \(\lambda_i\) falls at the end of \((v_i)\), then (5) drops out and we have again a contradiction. If \(l=0\), the sum of the inequalities in (4) gives at once the contradiction that the sum of the \(k\) integers \(\lambda_j\) is \(\geq mk + 2\), \(i \geq 2\).

**Editorial Note.** This interesting problem and its interesting solution were received from R. D. James with the following account of the origin of the problem. In an article by Heilbronn, Landau, and Sherk in the *Journal Tchécoslovaque de Mathématique et de Physique*, 65, 1935–36, pp. 117–140, there is a lemma (Satz 8) equivalent to the following: Given the numbers \(A_k\) defined as in (1) above but for \(m=2\), then \(A_{k+1}^2 \leq k^{2k}\). After a study of this it seemed to James that the result should be an actual equality, but he could not find a proof and suggested the problem to J. S. Vigder. The latter considered the more general problem using the positive integer \(m\) in place of 2, and saw that a proof involved the polynomial theorem as above, but he was unable to complete the proof and passed the matter on to the proposer. The proposer formulated the problem differently and came through with a solution resulting from his proof of his lemma (3).

**An Oval and its Normal Expansion**

4036 [1942, 340]. *Proposed by L. A. Santaló, Rosario, Argentina*

Let \(C_1\) be an oval with a continuously varying radius of curvature \(R\); at each point of \(C_1\) a normal of length \(R\) is drawn exteriorly giving points of a second curve \(C_2\) (which may not be convex); and let \(A\) be the area enclosed between the two curves. From a chosen fixed point a vector is drawn parallel to the normal at a point of \(C_1\) and of length \(R\) for that point, thus giving as the point varies on \(C_1\) a curve \(C_2\) having the area \(A_2\) and length \(L_2\). If \(L_2\) is the length of \(C_2\) and \(A_1\) is the area of \(C_1\), show that

(a) \(A = 3A_3\); \hspace{1cm} (b) \(L_2L_3 \geq 8\pi A_1\);

the equality in (b) is true only when \(C_1\) is a circle.

I. **Solution by Fritz John, University of Kentucky.** Let \(\rho(\alpha)\) denote the “function of support” of \(C_1\), i.e., \(\rho(\alpha)\) shall be the distance of that tangent of \(C_1\) from the origin, whose normal forms the angle \(\alpha\) with the \(x\)-axis (See Courant: Calculus, II, p. 213). Then

\[
x = \rho \cos \alpha - \rho' \sin \alpha, \quad y = \rho \sin \alpha + \rho' \cos \alpha
\]

is a parametric representation for \(C_1\). The radius of curvature of \(C_1\) is given by \(R = \rho + \rho''\), the enclosed area by

\[
A_1 = \frac{1}{2} \int_0^{2\pi} (xy' - yx')d\alpha = \frac{1}{2} \int_0^{2\pi} (\rho^2 + \rho \rho')d\alpha = \frac{1}{2} \int_0^{2\pi} (\rho^2 - \rho'^2)d\alpha.
\]
Similarly the parametric representations of $C_2$ and $C_3$ are respectively

\[ x = (2 \rho + \rho'') \cos \alpha - \rho' \sin \alpha, \quad y = (2 \rho + \rho'') \sin \alpha + \rho' \cos \alpha, \]

and

\[ x = (\rho + \rho'') \cos \alpha, \quad y = (\rho + \rho'') \sin \alpha; \]

hence the areas enclosed by $C_2$ and $C_3$ are easily found to be

\[ A_2 = \frac{1}{2} \int_0^{2\pi} (4\rho^2 - 7\rho'^2 + 3\rho''^2) d\alpha \]
\[ A_3 = \frac{1}{2} \int_0^{2\pi} (\rho^2 - 2\rho'^2 + \rho''^2) d\alpha. \]

Consequently $A = A_2 - A_1 = 3A_3$, which is the first statement.

Now "Wirtinger's inequality" states, that for a function $f(\alpha)$ of class $C'$ with $\int_0^{2\pi} f(\alpha)d\alpha = 0$

\[ \int_0^{2\pi} f''(\alpha)d\alpha > \int_0^{2\pi} f'(\alpha)d\alpha, \]

unless $f$ is of the form $f(\alpha) = a \cos \alpha + b \sin \alpha$; (see Hardy-Littlewood-Polya: Inequalities, pp. 185-187). For $f = \rho'$ it follows that $\int_0^{2\pi}\rho''(\alpha)d\alpha > \int_0^{2\pi}\rho'(\alpha)d\alpha$, and hence $A_3 > A_1$, unless $\rho' = a \cos \alpha + b \sin \alpha$; in the latter case $\rho = c + a \sin \alpha - b \cos \alpha$, and $C_1$ is a circle of radius $c$. The isoperimetric inequality (which may be based on Wirtinger's inequality), yields

\[ L_2 \geq 4\pi A_2, \quad L_3 \geq 4\pi A_3; \]

hence

\[ L_2 L_3 \geq 4\pi \sqrt{A_2 A_3} = 4\pi \sqrt{(3A_3 + A_1)A_3} > 4\pi \sqrt{4A_1^2} = 8\pi A_1 \]

unless $C_1$ is a circle.

In the case where $C_1$ is a circle of radius $c$, $C_2$ is a circle of radius $2c$, and $C_3$ a circle of radius $c$, so that $L_2 L_3 = 8\pi A_1$.

II. Solution by the Proposer. We consider two normals to $C_1$ corresponding to the directions $\phi$ and $\phi + d\phi$; a point on a normal whose distance to $C_1$ is a constant equal to $\lambda$ will describe a curve whose arc $s^*$ satisfies

\[ ds^* = (R + \lambda)d\phi. \]

The area $A$ will be then

\[ A = \int \int ds^* d\lambda = \int_0^{2\pi} d\phi \int_0^R (R + \lambda) d\lambda = \frac{3}{2} \int_0^{2\pi} R^2 d\phi = 3A_3, \]

which proves (a).

We have also, if $s_2$ is the arc of $C_2$,
where $R'$ represents the derivative with respect to $\phi$. We have also

$$ds_3 = \sqrt{R^2 + \frac{1}{R^2}} \, d\phi.$$  

From (1) and (2) we deduce, representing by $s_1$ the arc of $C_1$

$$ds_2 \geq 2Rd\phi = 2ds_1 \quad \text{and} \quad L_2 \geq 2L_1$$

$$ds_3 \geq Rd\phi = ds_1 \quad \text{and} \quad L_3 \geq L_1.$$

This gives us

$$L_2L_3 \geq 2L_1^2$$

But it is known that for every plane closed curve we have $L_1^2 - 4\pi A_1 \geq 0$; so this inequality and (3) proves the last part (b).

The equality in (b) is valid only if $R' = 0$, and then the radius of curvature is constant and the closed curve must be a circle.

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**NEWS AND NOTICES**

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. H. F. Bright of San Angelo College has been appointed to an assistant professorship at Denison University.

Dr. Jesse Douglas has been appointed to an assistant professorship at Brooklyn College.

Associate Professor R. C. Hildner of Mt. Union College has been appointed to an assistant professorship at the College of Wooster.

Professor E. J. Moulton of Northwestern University is on leave of absence and Professor H. T. Davis is acting head of the department of mathematics.

Assistant Professor W. H. Myers has been appointed acting head of the mathematics department at San José State College.

Mr. N. D. Nelson of the University of Wisconsin has been appointed to an assistant professorship at Amherst College.

Dr. E. A. Nordhaus of the University of Wisconsin has been appointed to an assistant professorship at Michigan State College.

Assistant Professor C. V. L. Smith of Lafayette College is now a lieutenant,