The Kinematic Formula in Integral Geometry for Cylinders.

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(Dedicated to Professor BENIAMINO SEGRE
on the occasion of his 70-th birthday)

Summary. – *We generalize the kinematic formula of Chern-Federer* (1.2) *to the case in which the moving manifold M^q is a cylinder in E^n. These cylinders and the corresponding kinematic density are suitable defined and some particular cases are considered in detail.*

1. – Introduction.

This paper will be concerned with the so called « kinematic formula » in Integral Geometry, due to FEDERER [2] and CHERN [1]. We shall refer mainly to the work of Chern, which likely assumes some more restrictive conditions than Federer, but remains into the mark of differential geometry. The approach of Federer is more in the mark of measure theory. The formula to which we refer is the following (CHERN [1]).

Let M^p, M^q be a pair of orientable, compact, differentiable manifolds (without boundary) of dimensions p, q immersed in euclidean space E^n. Let dg denote the kinematic density (= Haar measure of the group of motions in E^n) so normalized that the measure of all positions about a point is $O_{n-1} O_{n-2} \ldots O_1$ where

(1.1)
$$O_i = \frac{2\pi^{(i+1)/2}}{\Gamma((i+1)/2)}$$

is the volume of the i-dimensional unit sphere. Assume M^p fixed and M^q moving with the kinematic density dg. Let $\mu_e(X^k)$ $(0 < e < k)$ be the integral invariants (we call them Weyl's curvatures) of the riemannian k-dimensional manifold X^k to be defined below. Then the kinematic formula of Chern-Federer writes

(1.2)
$$\int \mu_e(M^p \cap gM^q) \, dg = \sum_{0 \leq i \leq e} c_i \mu_i(M^p) \mu_{e-i}(M^q) , \qquad i \text{ even}$$

where e even and $0 < e < p + q - n$. The integral on the left is over the whole

group of euclidean motions in E^n, i.e. over all positions of M^q, and $c_i = c_i(n, p, q, e)$ are numerical constants depending on n, p, q, e which may be calculated as follows. Put

$$(1.3) \qquad c_{e-i} = \frac{O_n \ldots O_1 O_{p+q-n+2} O_{q+1-i} O_{p+1-e+i}}{O_p O_{p+1} O_q O_{q+1} O_{p+q-n+2-i} O_{p+q-n+2-e+i}} \, b_{e,p+q-n+1-i}$$

where the b's are given by the following identity (with respect to the indeterminate x) with $m = p + q - n + 2$,

$$(1.4) \qquad \frac{2 O_{m-1} O_{m-2}}{O_{m-e-1}} x^{m-e-1} \sum \frac{(e/2)!}{(2\lambda)! \mu! (e/2 - 2\lambda - \mu)!} 2^{2\lambda} \frac{O_{2\lambda+m-e}}{O_{2\lambda}} x^{2\lambda+2\mu}$$
$$= b_{e,m-e-1} x^{m-e-1} + \ldots + b_{e,m-1} x^{m-1}$$

where the sum on the left side is over the following range of indices

$$0 < 2\lambda + \mu < e/2, \qquad 0 < \lambda, \mu.$$

For instance, for $i = 0$, (1.3) gives

$$(1.5) \qquad c_e = \frac{O_n \ldots O_1 O_{p+1-e}}{O_{p+1} O_p O_q O_{p+q-n+2-e}} \, b_{e,p+q-n+1}$$

and identifying the coefficients of x^{m-1} of both sides of (1.4) we have (since the relations $0 < 2\lambda + \mu < e/2$, $2\lambda + 2\mu = e$ give $\lambda = 0$, $\mu = e/2$)

$$b_{e,m-1} = \frac{O_{m-1} O_{m-2} O_{m-e}}{O_{m-e-1}}.$$

Hence

$$b_{e,p+q-n+1} = \frac{O_{p+q-n+1} O_{p+q-n} O_{p+q-n+2-e}}{O_{p+q-n+1-e}}$$

and thus

$$(1.6) \qquad c_e = \frac{O_n \ldots O_1 O_{p+1-e} O_{p+q-n+1} O_{p+q-n}}{O_{p+1} O_p O_q O_{p+q-n+1-e}}.$$

In particular we have

$$(1.7) \qquad c_0(n, p, q, 0) = \frac{O_n \ldots O_1 O_{p+q-n}}{O_p O_q}.$$

In like manner, if we put $i = e$ in (1.3), making use of (1.4), we easily get

$$(1.8) \qquad c_0(n, p, q, e) = \frac{O_n \ldots O_1 O_{q+1-e} O_{p+q-n+1} O_{p+q-n}}{O_p O_q O_{q+1} O_{p+q-n+1-e}}.$$

Our purpose is to extend (1.2) to the case in which M^e is a cylinder $Z_{h,m}$ in E^n. In this case, the kinematic density must be replaced by the density $dZ_{h,m}$ for cylinders, which we will define in section 3. The result is the formula (5.2) which contains as special or limiting cases many formulas in integral geometry in E^n. We consider with detail some of these particular formulas in section 6.

2. – The Weyl's curvatures.

We will define the curvatures $\mu_e(X^k)$ which appear in the kinematic formula (1.2) (see WEYL [6], CHERN [1], FEDERER [2]).

Let X^k be a differentiable riemannian manifold of dimension k and consider the classical differential forms ω_α, $\omega_{\alpha\beta}$ ($1 < \alpha, \beta, \gamma, \delta < k$) of the « moving frames » method, such that

(2.1) $\qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 \qquad d\omega_\alpha = \sum_\beta \omega_\beta \wedge \omega_{\beta\alpha} \qquad d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}$

where

(2.2) $\qquad\qquad\qquad \Omega_{\alpha\beta} = \tfrac{1}{2} \sum_{\gamma,\delta} S_{\alpha\beta\gamma\delta} \omega_\gamma \wedge \omega_\delta$

The coefficients $S_{\alpha\beta\gamma\delta}$ are essentially (though not exactly) the components of the Riemann-Christoffel tensor and have the same well known symmetry properties

(2.3) $\qquad \begin{cases} S_{\alpha\beta\gamma\delta} = - S_{\alpha\beta\delta\gamma} = - S_{\beta\alpha\gamma\delta} \\ S_{\alpha\beta\gamma\delta} = S_{\gamma\delta\alpha\beta} \,, \qquad S_{\alpha\beta\gamma\delta} + S_{\alpha\gamma\delta\beta} + S_{\alpha\delta\beta\gamma} = 0 \end{cases}$

Put

(2.4) $\qquad\qquad I_e = \dfrac{(-1)^{e/2}(k-e)!}{2^{e/2}k!} \sum \delta_{\beta_1\beta_2\cdots\beta_e}^{\alpha_1\alpha_2\cdots\alpha_e} S_{\alpha_1\alpha_2\beta_1\beta_2} \cdots S_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e}$

where e is an even integer satisfying $0 < e < k$ and $\delta_{\beta_1\cdots\beta_e}^{\alpha_1\cdots\alpha_e}$ is equal to $+1$ or -1 according as $\alpha_1, ..., \alpha_e$ is an even or odd permutation of $\beta_1, ..., \beta_e$ and is otherwise zero, and the summation is taken over all $\alpha_1, ..., \alpha_e$ and $\beta_1, ..., \beta_e$ independently from 1 to k. When X^k is oriented and compact, we let

(2.5) $\qquad\qquad\qquad \mu_e(X^k) = \int_{X^k} I_e \, d\sigma_k$

where $d\sigma_k$ is the volume element. This formula (2.5) defines the Weyl's curvatures (e even, $0 < e < k$). In particular we have

(2.6) $\qquad\qquad\qquad \mu_0(X^k) = \text{total volume of } X^k$

and, if k is even,

$$(2.7) \qquad \mu_k(X^k) = \tfrac{1}{2} \, O_k \, \chi(X^k)$$

where $\chi(X^k)$ denotes the Euler-Poincaré characteristic of X^k. (2.7) is the Gauss-Bonnet formula for compact even dimensional manifolds.

It would be of interest to compare these curvatures $\mu_e(X^k)$ with other curvatures which appear in the literature. For instance, if X^k is the boundary of a bounded convex set of E^{k+1} the volume $V(\varrho)$ of the parallel set to X^k at distance ϱ is (HADWIGER [3])

$$(2.8) \qquad V(\varrho) = \sum_{i=0}^{k+1} \binom{k+1}{i} W_i \varrho^i$$

and the volume of the « tube » at distance ϱ is

$$(2.9) \qquad V(\varrho) - V(-\varrho) = 2 \sum_{i=0}^{k+1} \binom{k+1}{i} W_i \varrho^i, \qquad i \text{ odd}.$$

The invariants $W_i(X^k)$ (quermassintegrale, introduced by Minkowski) may be written

$$(2.10) \qquad W_i = \frac{1}{k+1} \, M_{i-1}$$

where M_i $(i = 0, 1, 2, ..., k)$ are the i-th integrated mean curvatures

$$(2.11) \qquad M_i = \frac{1}{\binom{k+1}{i}} \int_{X_k} \left\{ \frac{1}{R_{s_1}} \cdots \frac{1}{R_{s_i}} \right\} d\sigma_k$$

where $d\sigma_k$ is the volume element of X^k and $\{1/R_{s_1}, ..., 1/R_{s_i}\}$ is the i-th elementary symmetric function of the principal curvatures of X^k. Comparing (2.9) with the Weyl's formula for the volume of tubes [6], we get

$$(2.12) \qquad \mu_e = M_e, \qquad e \text{ even}.$$

This formula holds for smooth compact hypersurfaces X^k of E^{k+1}, not necessarily convex. We deduce that, for e even, the mean curvatures (2.11) are isometric invariants of X^k which do not depend of its immersion in E^{k+1}, i.e. are intrinsic invariants.

3. – Density for cylinders.

Let M^h be an orientable, compact, differentiable manifold (without boundary) which belongs to a $(n-m)$-dimensional linear space E^{n-m} in E^n. Thus

$$(3.1) \qquad h + m < n.$$

Through each point $x \in M^h$ we consider the m-dimensional linear space E^m perpendicular to E^{n-m}. The set of all these E^m is called a cylinder $Z_{h,m}$ of dimension $h + m$, whose generators (or generating m-spaces) are the m-spaces E^m and which orthogonal cross section is the manifold M^h.

If we assume $Z_{h,m}$ moving in E^n, its position may determined by a $E^{n-m}(0)$ through a fixed point 0, orthogonal to the geneators E^m, and the position of the cross section M^h in $E^{n-m}(0)$. The density $dE^{n-m}(0)$ (= volume element of the grassmann manifold $G_{n-m,m}$ of all $(n - m)$-planes through 0 in E^n) and dg_{n-m} (kinematic density in E^{n-m}) are well known (see, for instance SANTALÓ [4], [5], CHERN [1], HADWIGER [3]). The density $dZ_{h,m}$ for cylinders $Z_{h,m}$ is then

(3.2)
$$dZ_{h,m} = dE^{n-m}(0) \wedge dg_{n-m} .$$

We recall these densities for completeness. If $(x; e_1, e_2, ..., e_n)$ is an orthogormal frame in E^n and we put

(3.3)
$$\omega_i = (dx \cdot e_i) , \quad \omega_{ih} = de_i \cdot e_h$$

then

(3.4)
$$dg_n = \Lambda \omega_i \Lambda \omega_{jh}$$

where the exterior products are between the ranges

(3.5)
$$i = 1, 2, ..., n ; \quad j = 2, 3, ..., n ; \quad k = 1, 2, ..., n - 1$$

with

$$j > k .$$

The differential form

(3.6)
$$d\sigma_n = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n$$

is the volume element in E^n. If E^{n-m} is spanned by $e_{m+1}, e_{m+2}, ..., e_n$ we have

(3.7)
$$dg_{n-m} = \omega_{m+1} \wedge ... \wedge \omega_n \Lambda \omega_{jk}$$

where

$$j = m + 2, ..., n ; \quad k = m + 1, ..., n - 1 , \quad k < j$$

and if E^m is spanned by $e_1, e_2, ..., e_m$ we have

(3.8)
$$dg_m = \omega_1 \wedge ... \wedge \omega_m \Lambda \omega_{jk}$$

where

$$j = 2, 3, ..., m ; \quad k = 1, 2, ..., m - 1 , \quad k < j .$$

Finally, assuming $E^{n-m}(0)$ parallel to E^{n-m}, we have

(3.9) $dE^{n-m}(0) = \Lambda \omega_{jk}$

where

$$j = m + 1, ..., n, \quad k = 1, 2, ..., m,$$

From (3.4), (3.2), (3.7) and (3.8) we deduce

(3.10) $dg = dg_n = dE^{n-m}(0) \wedge dg_{n-m} \wedge dg_m = dZ_{\lambda,m} \wedge dg_m .$

The exterior products in (3.8) and (3.7) have a clear geometrical meaning. Indeed we have

(3.11) $\omega_1 \wedge ... \wedge \omega_m = d\sigma_m =$ volume element in E^m ,

(3.12) $\omega_{m+1} \wedge ... \wedge \omega_n = d\sigma_{n-m} =$ volume element in E^{n-m} .

(3.13) $\Lambda \omega_{jk}(j = 2, ..., m; \ k = 1, ..., m-1; \ k < j) = dO_{m-1} \wedge dO_{m-2} \wedge ... \wedge dO_1$

where dO_i denotes the area element on the unit i-dimensional sphere in the space spanned by $e_1, e_2, ..., e_{i+1}$, and

(3.14) $\Lambda \omega_{jk}(j = m + 2, ..., n; \ k = m + 1, ..., n-1; \ k < j) = dO_{n-m-1} \wedge ... \wedge dO_1$

where dO_i is now the area element on the unit sphere in the $(i+1)$-space spanned by $e_{m+1}, ..., e_{m+i+1}$.

The density dE^m for the generating m-spaces E^m writes

(3.15) $dE^m = dE^{n-m}(0) \wedge d\sigma_{n-m}$

where $d\sigma_{n-m}$ is the element of volume in $E^{n-m}(0)$ at the intersection point $E^m \cap E^{n-m}(0)$. Having into account (3.2) and (3.7) we have

(3.16) $dZ_{\lambda,m} = dE^m \wedge dO_{n-m-1} \wedge ... \wedge dO_1 .$

We always consider the densities in absolute value, so that the order of the exterior products above is immaterial.

4. – The Weyl's curvatures for cylinders.

Choose the frame $(x; e_1, e_2, ..., e_n)$ such that $x \in Z_{\lambda,m}$ and $e_1, e_2, ..., e_{m+\lambda}$ spann the tangent space to $Z_{\lambda,m}$ in such a way that $e_1, e_2, ..., e_m$ spann the generator E_m

through x and $e_{m+1}, \ldots, e_{m+h}$ spann the tangent space to the cross section M^h. The volume elements in E^m, M^h and $Z_{h,m}$ are, respectively

(4.1) $d\sigma_m = \omega_1 \wedge \ldots \wedge \omega_m , \quad d\sigma_h = \omega_{m+1} \wedge \ldots \wedge \omega_{m+h} , \quad d\sigma_{m+h} = \omega_1 \wedge \ldots \wedge \omega_{m+h}$

and we have

(4.2) $d\sigma_{m+h} = d\sigma_m \wedge d\sigma_h .$

Since all E^m are perpendicular to E^{n-m}, we have $e_\alpha = $ constant for $\alpha = 1, 2, \ldots, m$ and thus $\omega_{\alpha k} = de_\alpha \cdot e_k = 0$ ($\alpha = 1, \ldots, m; \; k = 1, 2, \ldots, m + h$). Therefore $d\omega_{\alpha k} = \sum \omega_{\alpha i} \wedge \wedge \omega_{ik} + \Omega_{\alpha k} = 0$ and consequently $\Omega_{\alpha k} = 0$. Therefore, applying (2.2) to the cylinder $Z_{h,m}$ we have $S_{\alpha k i j} = 0$ ($\alpha = 1, 2, \ldots, m; \; k, i, j = 1, 2, \ldots, m + h$). According to the symmetry properties (2.3) of $S_{\alpha k i j}$ the equation $S_{\alpha k i j} = 0$ implies that $S_{k \alpha i j} = S_{k i \alpha j} = = S_{k i j \alpha} = 0$. The remaining $S_{k i j s}$ with $i, j, k, s = m + 1, m + 2, \ldots, m + h$ are the functions $S_{k i j s}$ corresponding to the cross section M^h.

Therefore, the sums on the right side of (2.4) are the same for $Z_{h,m}$ and for M^h and an easy calculation gives

(4.3)
$$I_e(Z_{h,m}) = \frac{\binom{h}{e}}{\binom{h+m}{e}} I_e(M^h) \qquad \text{if } e < h ,$$

$$I_e(Z_{h,m}) = 0 \qquad \text{if } e > h .$$

5. – The kinematic formula for cylinders.

Let E^m be a generator of $Z_{h,m}$ and consider a bounded domain $D^m \subset E^m$. Assume that $D^m = D^m(t)$ depends on a parameter t in such a way that $D^m \to E^m$ when $t \to \infty$. Consider the compact manifold $D^m \times M^h$. If $d\sigma_m$, $d\sigma_h$, $d\sigma_{h+m}$ denote respectively the volume elements in $D^m(t)$, M^h and $Z_{h,m}$ we have $d\sigma_{h+m} = d\sigma_m \wedge d\sigma_h$ and from (2.5) and (4.3) we get

(5.1)
$$\mu_e(Z_{h,m}) = \frac{\binom{h}{e}}{\binom{h+m}{e}} \mu_e(M^h)\sigma_m \qquad \text{if } e < h ,$$

$$\mu_e(Z_{h,m}) = 0 \qquad \text{if } e > h .$$

where σ_m denotes the volume of D^m.

We now apply formula (1.2) to M^s and $M^q = D^m \times M^h$. Using (3.10) having into account that $dg_m = d\sigma_m \wedge dO_{m-1} \wedge \ldots \wedge dO_1$ and making $t \to \infty$ (after division of

both sides by σ_m), so that $D^m \to E^m$ we get the desired formula

$$(5.2) \qquad \int\limits_{Z_{h,m} \cap M^p \neq \phi} \mu_e(M^p \cap Z_{h,m})\, dZ_{h,m} = \sum_{e-h \leqslant i \leqslant e} \frac{c_i}{O_1 O_2 \ldots O_{m-1}} \frac{\binom{h}{e-i}}{\binom{h+m}{e-i}} \mu_i(M^p)\mu_{e-i}(M^h)$$

$$(e \text{ even}, \quad 0 < e < p + h + m - n, \quad i > 0, \quad i \text{ even})$$

where $c_i = c_i(n, p, h + m, e)$ are the same constants as in Chern's formula (1.2).

6. – Particular cases.

1) Assume that $Z_{h,m}$ reduces to a m-plane E^m ($h = 0$). Then, according to (3.16) we have $dZ_{h,m} = dE^m \wedge dO_{n-m-1} \wedge \ldots \wedge dO_1$ and $\mu_0(M^0) = 1$. The sum on the right side of (5.2) reduces to the term $i = e$ and according to (1.6) we get (e even, $e < p + m - n$)

$$(6.1) \qquad \int\limits_{E^m \cap M^p \neq \phi} \mu_e(M^p \cap E^m)\, dE^m = \frac{O_{n-m} \ldots O_n\, O_{p+1-e} O_{p+m-n} O_{p+m-n+1}}{O_p O_{p+1} O_{p+m-n+1-e} O_1 \ldots O_m} \mu_e(M^p)\ .$$

This formula is due to CHERN [1]. For $p = n - 1$, see [4], [5].

2) Consider the case $e = p + m - n$. According to (2.7) we have

$$\mu_{p+m-n}(M^p \cap E^m) = \tfrac{1}{2} O_{p+m-n} \chi(M^p \cap E^m)$$

and (6.1) gives ($p + m - n$ even)

$$(6.2) \qquad \int\limits_{E^m \cap M^p \neq \phi} \chi(M^p \cap E^m)\, dE^m = \frac{2 O_{n-m} \ldots O_n O_{n-m+1} O_{p+m-n+1}}{O_1 O_p O_{p+1} O_1 \ldots O_m} \mu_{p+m-n}(M^p)\ .$$

3) The case $e = 0$. Applying (1.7), from (5.2) we deduce

$$(6.3) \qquad \int\limits_{Z_{h,m} \cap M^p \neq \phi} \mu_0(M^p \cap Z_{h,m})\, dZ_{h,m} = \frac{O_n \ldots O_m O_{p+h+m-n}}{O_p O_{h+m}} \mu_0(M^p)\mu_0(M^h)\ .$$

If $p + h + m - n = 0$, then μ_0 is equal to the number ν of intersection points of M^p and $Z_{h,m}$, and (6.3) gives

$$(6.4) \qquad \int \nu(M^p \cap Z_{h,m})\, dZ_{h,m} = \frac{2 O_n \ldots O_m}{O_p O_{n-p}} \mu_0(M^p)\mu_0(M^h)$$

and (6.1) gives

$$(6.5) \qquad \int v(M^p \cap E^m)\, dE^m = \frac{2O_{n-m}\ldots O_n}{O_p O_1 \ldots O_m}\, \mu_0(M^p)\,.$$

The integrals in (6.4) and (6.5) are extended over all positions of $Z_{h,m}$ and E^m, v being zero if they do not intersect the manifold M^p.

4) As last example, consider the case

$$p + h + m - n = 2\,, \qquad e = 2\,.$$

We have, by (2.7)

$$\mu_2(M^p \cap Z_{h,m}) = \tfrac{1}{2}O_2\,\chi(M^p \cap Z_{h,m})$$

From (1.3), (1.4), having into account that $q = h + m$, $p + q - n = 2$, we have

$$(6.6) \qquad c_2(n, p, h + m, 2) = \frac{O_n \ldots O_1\, O_{p-1}O_2 O_2}{O_p O_{p+1} O_{h+m} O_1}\,.$$

Using (6.6), (1.8) and the identity $O_i O_{i-2} = (i-1)O_i$, formula (5.2) gives

$$(6.7) \qquad \int\limits_{M^p \cap Z_{h,m} \neq \emptyset} \chi(M^p \cap Z_{h,m})\, dZ_{h,m} = \frac{2O_n \ldots O_m O_2}{O_p O_{h+m} O_1}\,.$$

$$\cdot \left[\frac{h(h-1)}{2\pi(h+m-1)}\mu_0(M^p)\mu_2(M^h) + \frac{O_{p-1}}{O_{p+1}}\mu_2(M^p)\mu_0(M^h) \right]\,.$$

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