AN INEQUALITY BETWEEN THE PARTS INTO WHICH
A CONVEX BODY IS DIVIDED BY A PLANE SECTION

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A new proof is given of an inequality of J. Bokowski and E. Sperner [1] referring to the product of the volume of the two parts into which a convex body is divided by a plane. The proof, which is given for dimensions \( n = 2, 3 \) uses known formulas of Integral Geometry and is generalized to convex bodies of the elliptic and hyperbolic spaces.

1. Introduction.

Let \( K \) be a convex domain in the euclidean \( n \)-space \( E_n \) and let \( L_{n-1} \) be an hyperplane which divides \( K \) into two parts \( K_1 \) and \( K_2 \). Let \( V(K_1), V(K_2) \) denote the volumes of \( K_1 \) and \( K_2 \) respectively, \( D \) the diameter of \( K \) and \( \sigma_{n-1} \) the \( (n-1) \)-dimensional volume of the intersection \( K \cap L_{n-1} \). Then, J. Bokowski and E. Sperner [1], [2] have proved the following inequality

\[
(1.1) \quad V(K_1) V(K_2) \leq \frac{(1-2^{-n})(n-1)\omega_{n-1}}{n(n+1)} D^{n+1} \sigma_{n-1}
\]

where \( \omega_{n-1} \) denotes the volume of the \( (n-1) \)-dimensional unit sphere. For \( n = 2, 3 \) this inequality takes the form

\[
(1.2) \quad F_1 F_2 \leq (D^4/4) \sigma_1, \quad V_1 V_2 \leq (7/48) \pi D^4 \sigma_2.
\]

Our purpose is to give a new proof of the particular cases (1.2) and to generalize these inequalities to the elliptic and hyperbolic spaces.
2. A fundamental Lemma.

Consider the segment $OA$ on the real line, of length $a$, and the segment $OX$ of length $x \leq a$. Let $f(r)$ be an integrable non-negative function defined on the closed interval $(0, a)$, which is strictly positive ($f(r) > 0$) for $0 < r < a$. Consider the integral

$$I(x) = \int f(t_2 - t_1) \, dt_1 \wedge dt_2, \quad t_1 \in OX, \quad t_2 \in XA.$$  

Then we have the following

**Lemma.** For any function $f(r)$ which satisfies the stated conditions, the integral (2.1) has its maximum for $x = a/2$.

**Proof.** Let $F(r)$ be a primitive of $f(r)$, with $r = t_2 - t_1$, and $G(r)$ a primitive of $F(r)$. We have

$$I(x) = \int_0^x [F(a - t_1) - F(x - t_1)] \, dt_1 =$$

$$= -G(a - x) + G(0) + G(a) - G(x).$$

In order that $I(x)$ have a maximum or minimum at the point $x$ we have $I'(x) = F(a - x) - F(x) = 0$ and since $F(x)$ is an increasing function we will have $a - x = x$ and $x = a/2$. This critical value $I(a/2)$ is a maximum because $I(0) = I(a) = 0$.

3. The case $n = 2$.

We want to consider separately the cases of the euclidean, elliptic and hyperbolic planes.

a) *The euclidean plane.* Consider the line $G_0$ which divides $K$ into two convex domains $K_1$ and $K_2$. Let $\sigma_1$ denote the length of the chord $G_0 \cap K$ (fig. 1). Consider the pair of points $P_1 \in K_1, P_2 \in K_2$ and the line $G$ determined by
them. It is well known the differential formula

\[(3.1) \quad d P_1 \wedge d P_2 = |t_2 - t_1| \ d G \wedge d t_1 \wedge d t_2\]

where \(d P_1, d P_2\) are the area elements of the plane at \(P_1, P_2\), \(d G\) is the density for lines on the plane and \(t_1, t_2\) are the abscissas of \(P_1, P_2\) on \(G\) [3, p. 28 and 46].

Integration of both sides of (3.1) over all pairs \(P_1 \in K_1, P_2 \in K_2\) gives: on the left side we get \(F_1 F_2\) and in the right side we have the integral (2.1) for the values

\[(3.2) \quad r = t_2 - t_1, \quad f = r, \quad F = (1/2) r^2, \quad G = (1/6) r^3.\]

Therefore, denoting by \(a\) the length of the chord \(G \cap K\), we get

\[(3.3) \quad I (x) = (1/2) a x (a - x), \quad I (a/2) = a^3/8.\]

Since the measure of the set of lines which cut the chord \(K \cap G_0\) is equal to \(2 \sigma_1\) and \(a \leq D\) \((D = \text{diameter of } K)\), we have

\[(3.4) \quad F_1 F_2 = \int I (x) \ d G \leq (D^3/4) \sigma_1\]

which is the first inequality (1.2)

b) The elliptic case. On the elliptic plane, instead of (3.1), we have [3, p. 316],

\[(3.5) \quad d P_1 \wedge d P_2 = \sin |t_2 - t_1| \ d G \wedge d t_1 \wedge d t_2.\]

We apply the fundamental lemma for the values

\[(3.6) \quad f = \sin r, \quad F = - \cos r, \quad G = - \sin r \]

and we have

\[(3.7) \quad I (x) = \sin (a - x) - \sin a + \sin x.\]

By integrating (3.5) over all pairs of points \(P_1 \in K_1, P_2 \in K_2\) we get

\[
F_1 F_2 = \int (\sin (a - x) - \sin a + \sin x) \ d G \leq \int (2 \sin (a/2) - \sin a) \ d G \\
= 4 \int \sin (a/2) \sin^2 (a/4) \ d G \leq 8 \sin (D/2) \sin^2 (D/4) \sigma_1.
\]
Therefore we have the following inequality

\[(3.8)\quad F_1 F_2 \leq 8 \sin (D/2) \sin^2 (D/4) \sigma_1.\]

We have applied that the measure of lines \(G\) which cut a segment of length \(\sigma_1\) is equal to \(2 \sigma_1\), the same that in the euclidean case [3, p. 310].

c) *The hyperbolic plane.* In this case, instead of (3.1) we have [3, p. 316]

\[(3.9)\quad d P_1 \wedge d P_2 = \sinh |t_2 - t_1| d G \wedge d t_1 \wedge d t_2.\]

In order to apply the lemma, we have

\[(3.10)\quad f = \sinh r, \quad F = \cosh r, \quad G = \sinh r\]

and therefore

\[(3.11)\quad I (x) = - \sinh (a - x) + \sinh a - \sinh x,\]

\[(3.12)\quad I (a/2) = 4 \sinh (a/2) \sinh^2 (a/4).\]

Since the measure of lines which intersect a segment of length \(\sigma_1\) is also \(2 \sigma_1\), [3, p. 310] we get the inequality

\[(3.12)\quad F_1 F_2 \leq 8 \sinh (D/2) \sin^2 (D/4) \sigma_1\]

which is the generalization to the hyperbolic plane of the first inequality of Bokowski-Sperner (1.2).

4. **The case** \(n = 3\).

We consider the three cases:

a) *Euclidean space.* With the customary notation we have [3, p. 237],

\[(4.1)\quad d P_1 \wedge d P_2 = (t_2 - t_1)^2 d G \wedge d t_1 \wedge d t_2.\]

By integration over all pairs \(P_1 \in K_1, P_2 \in K_2\), where \(K_1\) and \(K_2\) are now the bodies into which \(K\) is partitioned by the plane \(E_0\), calling \(V_1\) and \(V_2\) the volumes of these bodies, we have

\[(4.2)\quad f = (t_2 - t_1)^2 = r^2, \quad F = (1/3) r^1, \quad G = (1/12) r^0\]
and

\[ I(x) = (-1/12)(a - x)^4 + (1/12)a^4 - (1/12)x^4, \quad I(a/2) = (7/96)a^4. \]

Since the measure of the set of lines which cut the plane domain \( E_0 \cap K \) is \( \pi \sigma_2 \), where \( \sigma_2 \) denotes the surface area of \( E_0 \cap K \) [3, p. 233], we get

\[ (4.3) \quad V_1 V_2 \leq (7/96) \pi D^4 \sigma_2 \]

which is better than the second inequality of (1.2).

b) **Elliptic space.** In this case we have [3, p. 316]

\[ (4.4) \quad dP_1 \wedge dP_2 = \sin^2 (t_2 - t_1) dG \wedge dt_1 \wedge dt_2. \]

In order to apply the lemma, we have now

\[ (4.5) \quad f = \sin^2 r, \quad F = (1/2)(r - \sin r \cos r), \quad G = (1/4)(r' - \sin^2 r) \]

and according to (2.2) we have

\[ (4.6) \quad I(a/2) = (1/2)\sin^4 (a/2) + (1/8)(a^2 - \sin^2 a) \]

and since the measure of the lines which cut \( E_0 \cap K \) is equal to \( \pi \sigma_2 \) [3, p. 310], we get

\[ (4.7) \quad V_1 V_2 \leq (1/8)(4 \sin^4 (D/2) + D^2 - \sin^2 D) \pi \sigma_2 \]

which generalizes the inequality of Bokowski-Sperner to the elliptic space.

c) **Hyperbolic space.** In this case we have [3, p. 316]

\[ (4.8) \quad dP_1 \wedge dP_2 = \sinh^2 (t_2 - t_1) dG \wedge dt_1 \wedge dt_2 \]

and therefore we have, with the notations of n. 2,

\[ (4.9) \quad f = \sinh^2 r, \quad F = (1/2)(\sinh r \cosh r - r), \quad G = (1/4)(\sinh^2 r - r^2) \]

and thus

\[ I(x) = (1/4)(-\sinh^2 (a - x) + (a - x)^2 + \sinh^2 a - a^2 - \sinh^2 x + x^2), \]

\[ (4.10) \quad I(a/2) = (1/2) \sinh^4 (a/2) + (1/8)(\sinh^2 a - a^2). \]
Therefore, since the measure of the set of lines which intersect the set $E_0 \cap K$ is equal to $\pi \sigma_2$ [3, p. 310], we get

(4.11) \[ V_1 V_2 \leq (1/8) (4 \sinh^4 (D/2) + \sinh^2 D - D^3) \pi \sigma_2 \]

which is the generalization to the hyperbolic space of the second inequality (1.2).

5. A conjecture.

We have considered the case in which $K$ is partitioned by a line (for $n = 2$) or by a plane (for $n = 3$). More general is the case of a partition of $K$ into two sets $K_1, K_2$ not necessarily convex, separated by a curve (for $n = 2$) or by a surface (for $n = 3$). To apply the foregoing proof in this case we will need a lemma more general that the lemma stated in n. 2. We state it as the following conjecture:

Consider the closed interval $(0, a)$ on the real line, divided into $n + 1$ parts by the points $0 < a_1 < a_2 < \ldots < a_n < a$. Put $a_0 = 0$, $a_{n+1} = a$ and consider the sets of intervals

\[ T = \{(0, a_1), (a_2, a_1), (a_3, a_2), \ldots \} \]
\[ T^* = \{(a_1, a_2), (a_3, a_4), \ldots \}. \]

Consider the integral

(5.2) \[ I (a_1, a_2, \ldots, a_n, a) = \int_{T^*} \int_T f(|t - t^*|) dtdt*. \]

The conjecture is that this integral has a maximum for $n = 1$ and $a_i = a/2$ for any integrable and non-negative function defined on the interval $(0, a)$. If it is not true, seek additional conditions for $f$.

In order to apply this conjecture to the generalization of the inequality (1.1) to the elliptic and the hyperbolic spaces it should be sufficient to prove it for the cases $f = \sin^n r$ and $f = \sinh^n r$. 