ON PARALLEL HYPERSURFACES IN THE ELLIPTIC AND HYPERBOLIC n-DIMENSIONAL SPACE

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1. **Introduction.** Let S^{n-1} be a hypersurface of class C^3 in the elliptic or hyperbolic n-dimensional space, which is closed and bounding and whose principal curvatures with respect to an inside normal are all positive. Let $S^{n-1}(\lambda)$ be the hypersurface parallel to S^{n-1} at distance λ.

If $\rho_1, \rho_2, \cdots, \rho_{n-1}$ are the principal radii of curvature of S^{n-1} at a point P and dP denotes the element of area at P, the mean curvatures of S^{n-1} are defined by

$$(1.1) \qquad M_i = \int_{S^{n-1}} \left(\sum \frac{1}{\rho_{\nu_1}\rho_{\nu_2} \cdots \rho_{\nu_i}} \right) dP, \qquad i = 0, 1, \cdots, n-1,$$

where the sum is extended to the $C_{n-1,i}$ combinations of ith order of the indices $1, 2, \cdots, n-1$. In particular, M_0 coincides with the area A of S^{n-1}.

Herglotz [6][1] and, from a more general point of view, Allendoerfer [1] have obtained the area $A(\lambda)$ and volume $V(\lambda)$ of the parallel hypersurface $S^{n-1}(\lambda)$, which can be expressed as linear functions of the mean curvatures M_i of S^{n-1} with coefficients depending upon λ. For this purpose it is enough to find the expression of $A(\lambda)$, that is, $M_0(\lambda)$, because $V(\lambda)$ is then given by

$$(1.2) \qquad V(\lambda) = V + \int_0^\lambda A(\lambda) d\lambda.$$

The purpose of the present note is to extend these results to the evaluation of all mean curvatures $M_i(\lambda)$ of $S^{n-1}(\lambda)$. The resulting formulae are also linear with respect to M_i; they are (2.9) for the elliptic case, and (3.2) for the hyperbolic case. For $i=0$, they give the value of $A(\lambda)$ obtained by Herglotz and Allendoerfer.

As a consequence, in the elliptic case we obtain the relation (4.2) between the mean curvature of an S^{n-1} and those of its polar hypersurface. Finally we obtain the equations (5.3) which hold for the mean curvatures of convex surfaces of "constant width" in the elliptic or hyperbolic n-dimensional spaces.

In all these questions, in order to obtain simplifications in the re-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

sulting formulas the generalized Gauss-Bonnet formula as obtained by Allendoerfer-Weil [2] plays a fundamental role. In our particular case of the elliptic and hyperbolic space, this formula can be written (see [1]):

For $n-1$ even

$$C_{n-1}M_{n-1} + C_{n-3}M_{n-3} + \cdots + C_1M_1 + K^{n/2}V = -\omega^n\chi'/2 \tag{1.3}$$

and for $n-1$ odd

$$C_{n-1}M_{n-1} + C_{n-3}M_{n-3} + \cdots + C_0M_0 = \omega^n\chi'/2 \tag{1.4}$$

where ω^j is the surface area of a j-dimensional unit sphere ($\omega^0=2$) and

$$C_i = \frac{\omega^n}{\omega^i\omega^{n-1-i}} K^{(n-1-i)/2},$$

being $K=1$ in the elliptic and $K=-1$ in the hyperbolic case. χ' is the inner characteristic of the volume bounded by S^{n-1}; if S^{n-1} is a topologic sphere it is $\chi'=-1$ for $n-1$ even and $\chi'=1$ for $n-1$ odd.

2. **The elliptic case.** Let C_i ($i=1, 2, \cdots, n-1$) be the lines of curvature of S^{n-1} which pass through the point P and let ds_i be the element of arc of C_i at P. The element of area of S^{n-1} at P will be

$$dP = ds_1ds_2 \cdots ds_{n-1}. \tag{2.1}$$

If ρ_i is the principal radius of curvature at P corresponding to C_i and R_i represents the distance from P to the contact point of the normal to S^{n-1} at P with the envelope of the normals to S^{n-1} along C_i, the relation (see, for instance, [5, p. 214])

$$\rho_i = \tan R_i \tag{2.2}$$

is well known.

Furthermore if $d\alpha_i$ is the angle between two infinitely near normals to S^{n-1} along C_i at their intersection point,

$$ds_i = \sin R_i d\alpha_i. \tag{2.3}$$

From (2.1) and (2.3) we deduce

$$dP = \prod_{i=1}^{n-1} \sin R_i d\alpha_i. \tag{2.4}$$

Applying (2.4) to the hypersurface $S^{n-1}(\lambda)$, we have

$$dP(\lambda) = \prod_{i=1}^{n-1} \sin (R_i + \lambda) d\alpha_i \tag{2.5}$$

or, according to (2.4),

$$(2.6)\qquad \begin{aligned} dP(\lambda) &= \prod_{i=1}^{n-1} (\sin R_i \cos\lambda + \cos R_i \sin\lambda) d\alpha_i \\ &= \prod_{i=1}^{n-1} (\cos\lambda + \sin\lambda/\tan R_i) dP. \end{aligned}$$

From the definition (1.1) and from (2.2) we deduce

$$(2.7)\quad M_i(\lambda) = \int_{S^{n-1}(\lambda)} \left(\sum \frac{1}{\tan(R_{\nu_1}+\lambda)\cdots\tan(R_{\nu_i}+\lambda)} \right) dP(\lambda),$$

or, according to (2.5) and (2.4)

$$(2.8)\qquad \begin{aligned} M_i(\lambda) &= \int \sum \Bigg(\prod_{j=1}^{i} \cos(R_{\nu_j}+\lambda) \\ &\qquad\qquad \cdot \prod_{j=i+1}^{n-1} \sin(R_{\nu_j}+\lambda) \Bigg) d\alpha_1 \cdots d\alpha_{n-1} \\ &= \int_{S_{n-1}} \sum \Bigg(\prod_{j=1}^{i} \left(\frac{\cos\lambda}{\tan R_{\nu_j}} - \sin\lambda \right) \\ &\qquad\qquad \cdot \prod_{j=i+1}^{n-1} \left(\cos\lambda + \frac{\sin\lambda}{\tan R_{\nu_j}} \right) \Bigg) dP. \end{aligned}$$

The sums are always extended over all combinations of ith order of the indices $1, 2, \cdots, n-1$.

If we take into account (2.2) and the definition (1.1) of M_i, from the last equality results[2]

$$(2.9)\qquad M_i(\lambda) = \sum_{k=0}^{n-1} M_k \phi_{ik}(\lambda)$$

where

$$(2.10)\quad \phi_{ik}(\lambda) = \sum_{h=p}^{q} (-1)^{i-h} C_{n-1-k,i-h} C_{k,h} \sin^{i+k-2h}\lambda \cos^{n-1-i-k+2h}\lambda,$$

where the sum is extended over all values of h for which the combina-

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[2] The combinatory coefficients which appear in (2.10) are easily obtained if we observe that the number of terms in the sum (2.8) with k factors $1/\tan R_{\nu_j}$ and coefficient $\sin^{i+k-2h}\lambda \cos^{n-1-i-k+2h}\lambda$ is $C_{i,h} C_{n-1-i,k-h} C_{n-1,i}$ and the number of terms in the sum (1.1) which gives M_k is $C_{n-1,k}$. Therefore the product $M_k \sin^{i+k-2h}\lambda \cos^{n-1-i-k+2h}\lambda$ appears a number of times equal to the quotient of the two foregoing combinatory numbers, which is equal to $C_{n-1-k,i-h} C_{k,h}$.

tory symbols have a sense, that is

$$p = \max(0, i + k - n + 1), \qquad q = \min(i, k). \tag{2.11}$$

Formulas (2.9) and (2.10) solve our problem for the elliptic case.

3. **The hyperbolic case.** For the case of a hypersurface S^{n-1} in the hyperbolic n-dimensional space, formulas (2.2) and (2.3) must be replaced respectively by

$$\rho_i = \tanh R_i, \qquad ds_i = \sinh R_i d\alpha_i. \tag{3.1}$$

Exactly the same calculation as before gives now

$$M_i(\lambda) = \sum_{k=0}^{n-1} M_k \phi_{ik}(\lambda) \tag{3.2}$$

with

$$\phi_{ik}(\lambda) = \sum_{h=p}^{q} (-1)^{i-h} C_{n-1-k,i-h} C_{k,h} \sinh^{i+k-2h} \lambda \cosh^{n-1-i-k+2h} \lambda, \tag{3.3}$$

where p, q are given by (2.11).

4. **Polar surfaces.** In the elliptic case it is interesting to consider the polar surface $S^{n-1}(\pi/2)$ to the given S^{n-1}.

Applying (2.9), (2.10) for $\lambda = \pi/2$ we obtain

$$M_i(\pi/2) = (-1)^i M_{n-1-i}. \tag{4.1}$$

If M_i^P denotes the ith mean curvature of the polar surface, we have $M_i^P = (-1)^i M_i(\pi/2)$ and consequently

$$M_i^P = M_{n-1-i}. \tag{4.2}$$

For $i = 0$

$$A^P = M_{n-1},$$

which is a result due to Allendoerfer [1, formula (30)]. For $n = 3$ we get $A^P = M_2$, $M_1^P = M_1$ or, applying the Gauss-Bonnet formula (1.3)

$$M_1^P = M_1, \qquad A^P + A = -4\pi\chi'.$$

The last formula is due to Blaschke [4].

5. **Hypersurfaces of constant width.** Let us assume S^{n-1} to be a topological sphere such that the inward drawn normal at every point P cuts S^{n-1} beside P at only one opposite point P^*. Let Δ be the distance PP^* measured along the normal. If Δ is constant for every point

P, S^{n-1} is said to be a hypersurface of "constant width."

In such a case the normal at P^* coincides with P^*P. Indeed, if Q is a point of S^{n-1} such that the distance PQ is a maximum (P fixed, Q variable on S^{n-1}), QP must be normal to S^{n-1} at Q and therefore, by assumption, distance $QP=\Delta$; on the other hand, if P^*P were not normal to S^{n-1} at P^*, the distance PP^* would not be a maximum, thus distance $PP^*<$distance $PQ=\Delta$, contrary to the assumption.

Furthermore, according to the definition of the radii R_i and the assumption that they are not negative (see §1 and (2.2), (3.1)), the point of contact of the normal PP^* with the envelope of the normals along each line of curvature through P does lie inside the segment PP^*; therefore for the hypersurfaces of constant width, between the corresponding radii R_i, R_i^* at opposite points, the relation

$$R_i + R_i^* = \Delta, \qquad i = 1, 2, \cdots, n-1, \tag{5.1}$$

holds.

We have also $dP=(-1)^{n-1}dP^*$, and consequently (2.7) gives

$$M_i(-\Delta) = (-1)^{n-1-i}M_i \tag{5.2}$$

which holds the same in both elliptic and hyperbolic cases.

Therefore, taking into account the relations (2.9) and (3.2) we get: *Between the mean curvatures M_i of a hypersurface of constant width Δ in the elliptic or hyperbolic n-dimensional space, the relations*

$$M_i = (-1)^{n-1-i}\sum_{k=0}^{n-1} M_k\phi_{ik}(-\Delta), \qquad i = 0, 1, 2, \cdots, n-1, \tag{5.3}$$

hold, where ϕ_{ik} are given by (2.10) *in the elliptic case and by* (3.3) *in the hyperbolic case.*

Furthermore, if V is the volume enclosed by S^{n-1}, we have $V(-\Delta) =(-1)^nV$ and therefore (1.2) and (2.9), (3.2) give the following relation

$$V = (-1)^nV + (-1)^n\sum_{k=0}^{n-1} M_k\int_0^{-\Delta} \phi_{0k}(\lambda)d\lambda, \tag{5.4}$$

which must be added to the preceding ones (5.3).

The obtained relations (5.3) and (5.4) are, in general, not independent, as the following examples will show.

EXAMPLE 1. If $n=2$, (5.3) and (5.4) are equivalent to the unique relation

$$M_0 \sin \Delta - M_1(1 - \cos \Delta) = 0 \text{ (elliptic case)},$$

$$M_0 \sinh \Delta - M_1(1 - \cosh \Delta) = 0 \text{ (hyperbolic case).}$$

If L is the length and A the area enclosed by S^1, $M_0 = L$ and the Gauss-Bonnet formula gives $M_1 = 2\pi \pm A$; therefore the foregoing relations may be written respectively

$$(5.5) \qquad L = (2\pi - A) \tan (\Delta/2), \qquad L = (2\pi + A) \tanh (\Delta/2).$$

EXAMPLE 2. For $n=3$, if we set $M_0 = A$ and take into account (1.3) which gives $M_2 = 4\pi \pm A$, the relations (5.3) become equivalent to

$$(5.6) \qquad \begin{aligned} M_1 \cos \Delta &= 2(2\pi - A) \sin \Delta \text{ (elliptic case),} \\ M_1 \cosh \Delta &= 2(2\pi + A) \sinh \Delta \text{ (hyperbolic case).} \end{aligned}$$

(5.4) gives

$$2V = 2\pi\Delta - (M_1/2) \sin^2 \Delta - (2\pi - A) \sin \Delta \cos \Delta \text{ (elliptic case),}$$

$$2V = -2\pi\Delta - (M_1/2) \sinh^2 \Delta + (2\pi + A) \sinh \Delta \cosh \Delta \text{ (hyperbolic case).}$$

If we take into account (5.6), the last relations can be written respectively

$$(5.7) \qquad 4V = 4\pi\Delta - M_1, \qquad 4V = M_1 - 4\pi\Delta.$$

(5.5) and (5.7) are due to Blaschke [**3**]. For the analogous questions in the n-dimensional euclidean space, see [**7**].

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