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ON PERMANENT VECTOR-VARIETIES IN n DIMENSIONS *

BY L. A. SANTALÓ
Universidad de La Plata, Argentina

1. Introduction. J. L. SYNGE has recently given a generalization to tho euclidean space of n dimensions of ZORAWSKI's condition for the permanence of vector-lines in a moving fluid [2] (see also PRIM and TRUESDELL [1]).

The purpose of this note is to consider the more general case in which instead of vector-lines we have varieties of dimension $r \geq 1$ defined by certain vector fields. We obtain a necessary and sufficient condition for the permanence of these r-dimensional vector-varieties in a moving fluid. The method we follow is analogous to that of SYNGE.

We consider the euclidean n-space E_n with cartesian coordinates x^i. A set of r vectors $c_1, c_2, \cdots c_r$ define a r-vector

$$Y = [c_1, c_2, \cdots, c_r]$$

whose components are

$$Y^{i_1 i_2 \cdots i_r} = \begin{vmatrix} c_1^{i_1} & c_1^{i_2} \cdots & c_1^{i_r} \\ c_2^{i_1} & c_2^{i_2} \cdots & c_2^{i_r} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ c_r^{i_1} & c_r^{i_2} \cdots & c_r^{i_r} \end{vmatrix} \qquad (i_1, i_2, \cdots, i_r = 1, 2, \cdots, n).$$

We say that a multivector is equal zero, $Y=0$, when its components all vanish. The components of the vector c_α are denoted by $c_\alpha^i (i=1, 2, \cdots, n)$.

Throughout the paper greek indices are supposed to take the values $1, 2, \cdots, r$ and latin indices $1, 2, \cdots n$. The summation conventions is also used throughout the paper.

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2. The theorem. Let $c_\alpha(x, t)$, $v(x, t)$ be $r+1$ vector fields given in the euclidean n-space E_n depending upon the time t. The vector field v plays the part of velocity. Let c_α^i, v^i be the components of these vectors.

Let V_r be a r-dimensional variety which moves whit the fluid, that is, formed always of the same particles. Let

$$x = f(\theta^1, \theta^2, \cdots \theta^r, t) \tag{1}$$

be the parametric equation of V_r which depends upon t. The parameters θ^α remain constant as we follow a particle.

We have

$$\frac{\partial f}{\partial t} = v, \quad \frac{\partial f}{\partial \theta^\alpha} = \lambda_\alpha \tag{2}$$

where λ_α are r linearly independent vectors tangent to V_r.

If the vectors c_α at the points of V_r are all tangent to V_r we say that V_r is a vector-variety. The necessary and sufficient conditions that V_r should be a vector-variety at the time t are

$$\lambda_\alpha = A_{\alpha\beta}\, c_\beta \tag{3}$$

where $A_{\alpha\beta}$ are scalar factors (functions of x and t). Since the vectors c_α are assumed linearly independent and according to the definition (2) the vectors λ_α are also linearly independent, we have

$$det\,|\,A_{\alpha\beta}\,| \neq 0 . \tag{4}$$

If we define the multivectors

$$Y_\alpha = [\lambda_\alpha, c_1, c_2, \cdots c_r], \qquad \alpha = 1, 2, \cdots, r \tag{5}$$

the conditions (3) are equivalent to $Y_\alpha = 0$.

Following the way of SYNGE [2], we must examine how Y_α change as we move with the fluid, their rate of change being $\partial Y_\alpha / \partial t$.

We write

$$c_\alpha(x, t) = g_\alpha(\theta, t) \quad \text{when} \quad x = f(\theta^1, \theta^2, \cdots, \theta^r, t) \tag{6}$$

and

$$Y_\alpha(\theta, t) = [\lambda_\alpha, g_1, g_2, \cdots g_r]. \tag{7}$$

We have

$$\partial g_\alpha / \partial t = c_{\alpha, k}\, v^k + \partial c_\alpha / \partial t \tag{8}$$

$$\partial \lambda_\alpha / \partial t = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \theta^\alpha}\right) = \frac{\partial}{\partial \theta^\alpha} v = v_{,k}\, \lambda_\alpha^k \tag{9}$$

where the comma denotes partial differentiation.

Then from (7) and (5) we deduce

$$\frac{\partial Y_\alpha}{\partial t} = \left[\frac{\partial \lambda_\alpha}{\partial t}, c_1, c_2, \cdots, c_r\right] + \sum_\sigma \left[\lambda_\alpha, c_1, \cdots, c_{\sigma-1}, \frac{\partial g_\sigma}{\partial t}, c_{\sigma+1}, \cdots, c_r\right]$$

and according to (2), (3), (8) and (9),

$$\frac{\partial Y_\alpha}{\partial t} = [A_{\alpha\sigma} v_{,k} c^k_\sigma, c_1, \cdots, c_r] +$$

$$+ \sum_\sigma \left[A_{\alpha\beta} c_\beta, c_1, \cdots, c_{\sigma-1}, c_{\sigma,k} v^k + \frac{\partial c_\sigma}{\partial t}, c_{\sigma+1}, \cdots, c_r\right]$$

$$= [A_{\alpha\sigma} v_{,k} c^k_\sigma, c_1, \cdots, c_r] - A_{\alpha\sigma}\left[c_{\sigma,k} v^k + \frac{\partial c_\sigma}{\partial t}, c_1, c_2, \cdots, c_r\right]$$

$$= A_{\alpha\sigma}\left[v_{,k} c^k_\sigma - c_{\sigma,k} v^k - \frac{\partial c_\sigma}{\partial t}, c_1, \cdots, c_r\right] = A_{\alpha\sigma} Z_\sigma$$

where Z_σ is the $(r+1)$-vector

$$Z_\sigma = \left[v_{,k} c^k_\sigma - c_{\sigma,k} v^k - \frac{\partial c_\sigma}{\partial t}, c_1, \cdots, c_r\right]. \tag{10}$$

Having into account (4) the conditions $A_{\alpha\sigma} Z_\sigma = 0$ implie $Z_\sigma = 0$. Consequently $Z_\sigma = 0$ are necessary conditions for the permanence of the vector-varieties defined by the vector fields c_α. They are also snfficient, since if they hold the conditions $Y_\alpha = 0$ (necessary in order that the vector fields c_α define vector-varieties) implie $\partial Y_\alpha / \partial t = 0$.

Hence we have proved the following theorem

If r vector fields c_α define r-dimensional vector-varieties, then a necessary and sufficient condition for the permanence of these vector--varieties is that the multivectors (10) *shall be equal zero, that is,*

$$\left[v_{,k} c^k_\sigma - c_{\sigma,k} v^k - \frac{\partial c_\sigma}{\partial t}, c_1, \cdots, c_r\right] = 0 \quad (\sigma = 1, 2, \cdots, r).$$

For $r = 1$ we get the result of SYNGE.

REFERENCES

[1] R. PRIM and C. TRUESDELL, *A derivation of Zorawski's criterion for permanent vector-lines*, Proc. Amer. Math. Soc. **1**, 32-34 1950.

[2] J. L. SYNGE, *On permanent vector-lines in n dimensions*, Proc. Amer. Math. Soc. **2**, 370-372 (1951).