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INTEGRAL GEOMETRY IN PROJECTIVE AND AFFINE SPACES

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Introduction

The fundamental concepts of integral geometry in a general space in which a transitive Lie group of automorphisms has been defined were given by Chern [3]. Our purpose is to apply these fundamental concepts in order to study the integral geometry in projective and affine spaces and to deduce from this study some geometrical consequences.

In §1 we give a brief summary of the ideas of Chern, who was the first to apply the Cartan's theory of Lie groups to the integral geometry; only the condition (1.11) is slightly different from that given by Chern. In §2 we study the integral geometry in projective space. In §3 we consider the unimodular center affine group and its integral geometry and in §4 we apply the obtained results in order to give an elementary proof of a theorem of Minkowski-Hlawka belonging to the geometry of numbers. Finally, in §5, we indicate the main results of integral geometry in unimodular affine space.

1. On the measure of sets of geometrical elements with respect to a given group of Lie

Let E be a space of points x in which an r-parameter Lie group G_r of automorphisms has been defined. Let $a^1, a^2, \cdots, a^2$ be the parameters of G_r and

$$(1.1) \qquad \omega_1(a, da), \qquad \omega_2(a, da), \cdots, \omega_r(a, da)$$

its relative components (see Cartan [1] p. 79), that is, a set of r linearly independent Pfaffian forms invariant under the first group of parameters of G_r. These relative components satisfy the equations of structure of Maurer-Cartan

$$(1.2) \qquad \omega_i' = \sum_{j,k=1}^{r} c_{jk}^i[\omega_j\omega_k], \qquad i = 1, 2, \cdots, r$$

where c_{jk}^i are the constants of structure of G_r, and ω_i' denotes the exterior derivative of the form ω_i. The square brackets denote exterior multiplication.

Let H be a geometrical element of E depending upon h parameters. By a geometrical element we understand a set of points of E which may be determined by a finite number of parameters. For instance, if E is the 3-dimensional euclidean space, points, straight lines, quadrics, are geometrical elements. In general, any figure transformed of a given fixed figure F by $s \in G_r$ is a geometrical element, because it may be determined by the parameters of s.

Let us assume that the subgroup of G_r, which leaves invariant the geometrical element H, is a continuous subgroup g_{r-h} depending upon $r - h$ parameters. In the space of parameters, g_{r-h} will be represented by an $(r - h)$-dimensional variety which we shall represent by the same notation g_{r-h}. The variety g_{r-h} and its transformed sg_{r-h}, by the operations s of the group of parameters of G_r, fill up the whole space of parameters and have the property that no two of them can have common point without being identical. Thus the varieties sg_{r-h} are the integral varieties of a completely integrable Pfaffian system. Furthermore the totality of varieties sg_{r-h} is invariant with respect to the first group of parameters; consequently the left hand sides of the Pfaffian system are linear combinations with constant coefficients of the relative components $\omega_1, \cdots, \omega_r$. Since the relative components are determined up to a linear transformation with constant coefficients, we may suppose the Pfaffian system which integral varieties are sg_{r-h} to be

$$(1.3) \qquad \omega_1 = 0, \qquad \omega_2 = 0, \cdots, \omega_h = 0.$$

Let

$$(1.4) \qquad \varphi_1(a^1, \cdots, a^2) = \alpha^1, \cdots, \varphi_h(a^1, \cdots, a^r) = \alpha^h$$

be h independent first integrals of (1.3). That means that to each set of constants $\alpha^1, \cdots, \alpha^h$ corresponds an integral variety sg_{r-h}. Thus we can take $\alpha^1, \cdots, \alpha^h$ as coordinates of the variety sg_{r-h}. In the original space E, to each sg_{r-h} corresponds a geometrical element sH, transformed of H by s. Conversely, to each sH of E corresponds a variety sg_{r-h}, consequently $\alpha^1, \cdots, \alpha^h$ may also be considered as coordinates of the geometrical element sH.

By "density" of the elements sH we shall mean an exterior differential form of order h of the form

$$(1.5) \qquad dH = f(\alpha^1, \cdots, \alpha^h)\,[d\alpha^1\, d\alpha^2 \cdots d\alpha^h]$$

such that its value be invariant under the group G_r, i.e., under the first group of parameters. Since G_r is transitive with respect to the elements sH, this density, if it exists, is unique up to a constant factor. The measure of a set of elements sH will then be the integral of dH extended over the set.

Being independent first integrals of (1.3), the differentials $d\alpha^i$ are linearly independent combinations of $\omega_1, \cdots, \omega_h$ and we get, by exterior multiplication, an expression of the form

$$(1.6) \qquad [d\alpha^1 \cdots d\alpha^h] = \Delta(a^1, \cdots, a^2)[\omega_1 \cdots \omega_h], \qquad \Delta \neq 0$$

or

$$(1.7) \qquad [\omega_1 \cdots \omega_h] = (1/\Delta)[d\alpha^1 \cdots d\alpha^h].$$

The left hand side is invariant under the first group of parameters; therefore, in order that (1.5) be a density, we must have, up to a constant factor, $f = 1/\Delta(a)$, that is, $\Delta(a)$ must be a function of $\alpha^1, \cdots, \alpha^h$ only (condition of Chern [3]). The density is then defined up to a constant factor by

$$(1.8) \qquad dH = [\omega_1\omega_2 \cdots \omega_h].$$

The foregoing condition of Chern is, in general, not easy to apply. In many cases it is more convenient to express it in the following equivalent form.

The exterior differential form $[\omega_1 \cdots \omega_h]$ is always invariant under the first group of parameters. However, it is not always a density because it depends on the r parameters a^i and, though each point a determines an element sH, to each sH corresponds all points a of the corresponding variety sg_{r-h}. In order that (1.8) be a density its value does not change when the points are displaced on the varieties sg_{r-h}. That is, if we consider an h-dimensional variety V_h in the space of parameters, in order that (1.8) be a density, it is necessary and sufficient that the integral

$$\int_{V_h} [\omega_1 \cdots \omega_h] \tag{1.9}$$

be invariant when each point of V_h displaces on the variety sg_{r-h} which passes through it. That is equivalent to saying that the integral (1.9) is zero when extended over any closed h-dimensional variety (observe that a general V_h intersects the varieties sg_{r-h} in points only). The generalized Stokes formula (see, for instance [2] p. 40) says that if V_{h+1} is any $(h + 1)$-dimensional domain and ∂V_{h+1} is its boundary, it is

$$\int_{\partial V_{h+1}} [\omega_1 \cdots \omega_h] = \int_{V_{h+1}} [\omega_1 \cdots \omega_h]'. \tag{1.10}$$

Since ∂V_{h+1} is closed and of dimension h, the last integral must be zero for any integration domain V_{h+1}; consequently the integrand must be zero, and we get:

A necessary and sufficient condition for (1.8) *to be a density for the elements sH is*

$$[\omega_1 \cdots \omega_h]' = 0. \tag{1.11}$$

As a first and immediate application of this result, let us consider the case in which H is a geometrical element such that the subgroup of G_r, which leaves it invariant, reduces to the identity. In this case H depends on r parameters and to each element sH corresponds an unique transformation s of the group of parameters; the varieties sg_{r-h} are the points of the space of parameters and the density (1.8) is formed by the exterior product of all relative components ω_i. Having taken into account the equations of structure (1.2) the condition (1.11) is obviously satisfied in this case. We shall write

$$dG_r = [\omega_1 \cdots \omega_r] \tag{1.12}$$

and following Poincaré and Blaschke the density dG_r will be called the "cinematic density" of the group G_r. Thus we have: *the cinematic density of a group always exists.*

Let us now consider the case in which the subgroup g_{r-h}, which leaves H invariant, is a discrete group. In this case g_{r-h} will be represented in the space of parameters by a set of $(r - h)$-dimensional varieties, without common point, and congruent with respect to g_{r-h}. Analogously as in the continuous case, each of these varieties will be represented by a Pfaffian system of the form (1.3). The density for sets of elements sH will then be $dH = [\omega_1 \cdots \omega_h]$, assuming that (1.11) is satisfied. The only thing we have into account is that in the present case to each partial variety which composes sg_{r-h} corresponds the same geometrical element sH, so that, in order to measure a set of different elements sH, the domain of integration must be considered in the space of the factor group G_r/g_{r-h}.

In §4 we shall see an example in which g_{r-h} is discrete.

2. Integral Geometry in Projective Space

Let P_n be the real n-dimensional projective space. Following E. Cartan, [1] p. 75, a set of $n + 1$ real numbers $x^0, \cdots, x^n$ (not all equal to zero) will be called an "analytic point", represented by x, whose coordinates are the numbers x^i. To each analytic point x corresponds the "geometric point" whose homogenous coordinates are x^i.

The projective group G_r ($r = n(n + 2)$), which we shall denote by $\mathfrak{P}$, may be represented by

$$(x^k)' = \sum_{i=0}^{n} a_i^k x^i \qquad (k = 0, 1, \cdots, n) \tag{2.1}$$

with the condition

$$| a_i^k | = 1. \tag{2.2}$$

That means that each projective transformation is determined by $n + 1$ analytic points $a_i(a_i^0, \cdots, a_i^n)$, $i = 0, 1, \cdots, n$ which satisfy the condition (2.2). Instead of (2.2) it will be more convenient to write

$$| a_0 a_1 \cdots, a_n | = 1 \tag{2.3}$$

where the left hand side represents the determinant formed with the coordinates of the analytic points a_i.

The $r = n(n + 2)$ relative components of the projective group are defined by the equations (Cartan [1] p. 84)

$$da_i = \sum_{k=0}^{n} \omega_{ik} a_k, \qquad i = 0, 1, \cdots, n. \tag{2.4}$$

From (2.3) and (2.4) we deduce

$$\omega_{ik} = | a_0 \cdot a_1 \cdots a_{k-1} da_i a_{k+1} \cdots a_n | \tag{2.5}$$

with the condition, obtained by differentiation of (2.3),

$$\sum_{i=0}^{n} \omega_{ii} = 0. \tag{2.6}$$

The equations of structure are obtained by taking the exterior derivative of (2.4) and taking into account the relations (2.4) themselves. The result is

$$(\omega_{ij})' = \sum_{k=0}^{n} [\omega_{ik} \omega_{kj}]. \tag{2.7}$$

We have now all the necessary elements to study the integral geometry in P_n.

Let us consider the geometrical elements sH defined by a completely integrable Pfaffian system

$$\omega_{i_1 j_1} = 0, \qquad \omega_{i_2 j_2} = 0, \cdots, \omega_{i_h j_h} = 0. \tag{2.8}$$

This is the system (1.3) of §1, which means that the integral varieties sg_{r-h} of (2.8) correspond to the geometrical elements sH obtained applying a projective transformation to one, say H, of them.

A necessary and sufficient condition in order that the elements sH possess an invariant density is given by (1.11); that is

$$[\omega_{i_1 j_1} \cdots \omega_{i_h j_h}]' = 0. \tag{2.9}$$

We want to see the form which takes this condition in the particular case of the projective group. We have, according to (2.7),

$$[\omega_{i_1 j_1} \cdots \omega_{i_h j_h}]' = \sum_{m=1}^{h} (-1)^{m-1} [\omega_{i_1 j_1} \cdots \omega_{i_{m-1} j_{m-1}} \sum_{l=0}^{n} [\omega_{i_m l} \omega_{l j_m}] \omega_{i_{m+1} j_{m+1}} \cdots \omega_{i_h j_h}].$$

Since the system (2.8) is completely integrable, the theorem of Frobenius, [1] p. 193, says that at least one of the forms $\omega_{i_m l}$, $\omega_{l j_m}$ for any m and l belongs to the set (2.8). Therefore it is

$$[\omega_{i_1 j_1} \cdots \omega_{i_h j_h}]' = (-1)^h [\omega_{i_1 j_1} \cdots \omega_{i_h j_h} \sum_{m=1}^{h} (\omega_{j_m j_m} - \omega_{i_m i_m})]. \tag{2.10}$$

Thus, if we set $dH = [\omega_{i_1 j_1} \cdots \omega_{i_h j_h}]$, we have the following

LEMMA 2.1. *In order that the geometrical elements defined by the system* (2.8) *possess an invariant density with respect to the projective group, it is necessary and sufficient that*

$$\left[dH \cdot \sum_{m=1}^{h} (\omega_{j_m j_m} - \omega_{i_m i_m}) \right] = 0. \tag{2.11}$$

Let us apply this lemma in order to see if the linear subspaces S_h of dimension h ($h < n$) possess a density. The geometrical element H is now a particular S_h and the set of geometrical elements sH is the set of all h-dimensional subspaces in P_n, since each of them may be obtained from S_h by a suitable projectivity. Consider the S_h defined by the analytic points $a_0, \cdots, a_h$, that is, the linear subspace defined by the parametric equations

$$x^i = \sum_{k=0}^{h} \lambda^k a_k^i, \qquad i = 0, 1, \cdots, n. \tag{2.12}$$

The subgroup g of projectivities which leave this S_h invariant is characterized by the condition that the points $a_0, \cdots, a_h$ remain in S_h. Consequently the differentials da_i ($0 \leqq i \leqq h$) must be linear combinations of $a_0, \cdots, a_h$ and (2.4) gives

$$\omega_{ik} = 0 \quad \text{for} \quad \begin{cases} i = 0, 1, \cdots, h \\ k = h + 1, \cdots, n. \end{cases} \tag{2.13}$$

This is the Pfaffian system which corresponds to the linear subspaces S_h; it is composed of $(h + 1)(n - h)$ equations. According to the lemma, a density for S_h will exist if and only if

$$\left[\prod_{\substack{i=0,\cdots,h\\k=h+1,\cdots,n}} \omega_{ik} \left(\sum_{l=h+1}^{n} \omega_{ll} - \sum_{l=0}^{h} \omega_{ll}\right)\right] = 0. \tag{2.14}$$

From (2.6) we deduce

$$\sum_{l=h+1}^{n} \omega_{ll} = -\sum_{l=0}^{h} \omega_{ll}.$$

Consequently the left hand side of (2.14) takes the form

$$-2\left[\prod_{\substack{i=0,\cdots,h\\k=h+1,\cdots,n}} \omega_{ik} \cdot \sum_{l=0}^{h} \omega_{ll}\right]. \tag{2.15}$$

Since between the relative components ω_{ik} there exists only the relation (2.14), the exterior product (2.15) cannot be zero. Consequently we have:

THEOREM 2.1. *The linear subspaces have no invariant density with respect to the projective group.*

In other words: it is not possible to define a measure, given by an integral of a form like (1.5), for sets of linear subspaces which will be invariant under the projective group.

Let us now consider as geometrical elements sets of linear subspaces $S_{h_1} + S_{h_2} + \cdots + S_{h_m}$, without common point and satisfying the condition

$$h_1 + h_2 + \cdots + h_m + m \leqq n + 1. \tag{2.16}$$

We may take

$$\begin{array}{llll}
 & S_{h_1} & \text{defined by the analytic points} & a_0, a_1, \cdots, a_{h_1} \\
 & S_{h_2} & \text{“} \qquad \text{“} & a_{h_1+1}, \cdots, a_{h_1+h_2+1} \\
(2.17) & S_{h_3} & \text{“} \qquad \text{“} & a_{h_1+h_2+2}, \cdots, a_{h_1+h_2+h_3+2} \\
 & \cdots\cdots\cdots & & \\
 & S_{h_m} & \text{“} \qquad \text{“} & a_{h_1+\cdots+h_{m-1}+m-1}, \cdots, a_{h_1+\cdots+h_m+m-1}.
\end{array}$$

The subgroup of projectivities which leave $S_{h_1} + S_{h_2} + \cdots + S_{h_m}$ invariant is characterized by the condition that the differentials da_{i_s} are linear combinations of a_{i_s} for $h_1 + \cdots + h_s + s \leqq i_s \leqq h_1 + \cdots + h_{s+1} + s$ and $0 \leqq s \leqq m - 1$. Consequently, according to (2.4) we must have $\omega_{ij} = 0$ for all pairs i, j between the limits

$$\begin{gathered}
0 \leqq i \leqq h_1, \qquad h_1 + 1 \leqq j \leqq n; \\
h_1 + 1 \leqq i \leqq h_1 + h_2 + 1, \qquad 0 \leqq j \leqq h_1, \qquad h_1 + h_2 + 2 \leqq j \leqq n; \\
\cdots\cdots\cdots\cdots \\
h_1 + \cdots + h_{m-1} + m - 1 \leqq i \leqq h_1 + \cdots + h_m + m - 1, \\
0 \leqq j \leqq h_1 + \cdots + h_{m-1} + m - 2, \qquad h_1 + \cdots + h_m + m \leqq j \leqq n
\end{gathered} \tag{2.18}$$

In order to apply the lemma, we observe that, having taken into account (2.5), for each row of (2.17) the sum of ω_{jj} is equal, up to the sign, to the corresponding sum of ω_{ii}. Consequently condition (2.11) is written

$$\left[\prod \omega_{ij} \cdot \sum_{l=0}^{h_1+\cdots+h_m+m-1} \omega_{ll}\right] = 0 \tag{2.19}$$

where Π is extended over all ω_{ij} with the conditions (2.18). Since $i \neq j$ in this product, we find that (2.19) holds only if

$$\sum_{l=0}^{h_1+\cdots+h_m+m-1} \omega_{ll} = 0. \tag{2.20}$$

Since (2.6) is the only relation between the relative components ω_{ij}, (2.20) holds only if

$$h_1 + h_2 + \cdots + h_m + m = n + 1. \tag{2.21}$$

Then we have:

THEOREM 2.2. *In order that the elements* $(S_{h_1} + S_{h_2} + \cdots + S_{h_m})$ *composed by* m *linear subspaces of dimensions* h_i, *without common point and satisfying the relation* (2.16), *have an invariant density with respect to the projective group, it is necessary and sufficient that the relation* (2.21) *holds. In this case, assumed each* S_{h_i} *defined by the points indicated in* (2.17), *the density is given by the exterior product of the forms* ω_{ij} *(given by* (2.5)) *where* i, j *are submitted to the conditions* (2.18).

For instance, on the straight line, $n = 1$, the pairs of points have a density with respect to the projective group, which is given by $(\xi - \eta)^{-2}[d\xi d\eta]$, where ξ, η are the non-homogenous coordinates of the points.

For the case $m = 2$, the foregoing result, following a completely different way, was obtained by Varga [7].

Finally let us observe that the cinematic density for the projective group $\mathfrak{P}$ is expressed by the exterior product of all independent ω_{ij}, given by (2.5), that is

$$d\mathfrak{P} = \left[\prod_{i,j=0}^{n}{}' \omega_{ij}\right] \tag{2.22}$$

where the accent denotes that the factor ω_{nn} is excluded, because according to (2.6) it is not independent from the others.

Taking into account (2.5) and setting

$$dA_i = [da_i^0 da_i^1 \cdots da_i^n] \qquad i = 0, 1, \cdots, n \tag{2.23}$$

we may also write, after applying a known property about adjoint determinants, [5] p. 78.

$$d\mathfrak{P} = \left[\prod_{i=0}^{n-1} dA_i \cdot \sum_{j=0}^{n} (-1)^j a_n^j \, da_n^0 \cdots da_n^{j-1} \, da_n^{j+1} \cdots da_n^n\right]. \tag{2.24}$$

That is: the measure of a set of projectivities defined by the analytic points a_i ($i = 0, 1, \cdots, n$) with the condition (2.3), is given, up to a constant factor, by the integral of the exterior differential form (2.24) extended over the set.

3. The Group of Unimodular Center-Affine Transformations

Let us now consider the n-dimensional affine space and in it the group $\mathfrak{A}$ of affine transformations of modulo 1, which leaves invariant the origin $O(0, 0, \cdots, 0)$ (unimodular center-affine group). If $x^1, x^2, \cdots, x^n$ are the non-homogeneous coordinates of the point x and $a_i^1, a_i^2, \cdots, a_i^n$ those of the points a_i ($i = 1, 2, \cdots, n$)which determine the center-affine transformation, the equations of the group may be written

$$(x^i)' = \sum_{k=1}^{n} a_k^i x^k, \qquad i = 1, 2, \cdots, n \tag{3.1}$$

with the condition

$$| a_1 a_2 \cdots a_n | = 1. \tag{3.2}$$

In the present section there is no more distinction between analytic and geometric points; since the coordinates are non-homogeneous, all points are geometric ones.

Analogously to the case of the projective group, the relative components are now defined by the equations

$$da_i = \sum_{j=1}^{n} \omega_{ij} a_j, \qquad i = 1, 2, \cdots, n \tag{3.3}$$

which, having into account (3.2) give

$$\omega_{ij} = | a_1 a_2 \cdots a_{j-1} da_i da_{j+1} \cdots a_n | \tag{3.4}$$

with the relation

$$\sum_{i=1}^{n} \omega_{ii} = 0. \tag{3.5}$$

The equations of structure are

$$(\omega_{ij})' = \sum_{k=1}^{n} [\omega_{ik}\, \omega_{kj}]. \tag{3.6}$$

For the linear subspaces which pass through the origin o, the group $\mathfrak{A}$ in the n-dimensional space, coincides with the projective group in the $(n - 1)$-dimensional space, assuming each S_h in the affine space as equivalent to a S_{h-1} in the projective space. Consequently, Theorems 2.1 and 2.2 may now be announced:

THEOREM 3.1. *The linear subspaces which pass through the origin have no invariant density (or measure) with respect to the unimodular center-affine group A. The elements $(S_{h_1} + S_{h_2} + \cdots + S_{h_m})$ composed of m linear subspaces of dimension*

h_i, passing through the origin and having no other common point, have an invariant density with respect to $\mathfrak{A}$ if and only if the condition

$$h_1 + h_2 + \cdots + h_m = n \tag{3.7}$$

holds.

In the last case, if each S_{h_i} is determined by the points $a_{h_1+\cdots+h_{i-1}+1}$, $a_{h_1+\cdots+h_{i-1}+2}, \cdots, \cdots, a_{h_1+h_2+\cdots+h_i}$, the density for sets of elements $(S_{h_1} + S_{h_2} + \cdots + S_{h_m})$ is given by the exterior product of the differential forms ω_{ij}, given by (3.4), corresponding to the values i, j between the limits

$$\begin{aligned} &h_1 + \cdots + h_{i-1} + 1 \leqq i \leqq h_1 + \cdots + h_i, \\ &1 \leqq j \leqq h_1 + \cdots + h_{i-1}, \qquad h_1 + \cdots + h_i + 1 \leqq j \leqq n \end{aligned} \tag{3.8}$$

for $i = 1, 2, \cdots, m$.

EXAMPLE. On the plane, $n = 2$, it is not possible to define an invariant measure with respect to $\mathfrak{A}$ for sets of straight lines through the origin. However this measure exists for sets of pairs of straight lines, because (3.7) holds for $h_1 = h_2 = 1$. If the pair of straight lines is determined by the angles φ_1, φ_2 which they form with the x-axis, it is easily found that the density takes the value $d(S_1 + S_2) = \sin^{-2}(\varphi_2 - \varphi_1)[d\varphi_1 d\varphi_2]$.

Let us now see if density exists for linear subspaces S_h which do not pass through the origin. We may consider as fixed subspace S_h, that which contains the point a_1 and is parallel to oa_i $(i = 2, 3, \cdots, h + 1)$. If this S_h is assumed fixed, the differentials da_i $(i = 1, 2, \cdots, h + 1)$ must be linear combinations of $a_2, \cdots a_{h+1}$. Consequently, (3.3) gives

$$\begin{aligned} &\omega_{i1} = 0 \qquad \text{for} \qquad i = 1, 2, \cdots, h + 1 \\ &\omega_{ij} = 0 \qquad \text{for} \qquad i = 1, \cdots, h + 1, j = h + 2, \cdots, n. \end{aligned} \tag{3.9}$$

According to §1, in order that the sets of S_h have a density we must have

$$\left[\prod_{i=1,\cdots,h+1} \omega_{i1} \cdot \prod_{\substack{i=1,\cdots,h+1 \\ j=h+2,\cdots,n}} \omega_{ij}\right]' = 0, \tag{3.10}$$

or, according to the Lemma 2.1, which is also applicable in this case because the equations of structure (2.7) and (3.6) have the same form in both cases, we have

$$\left[\prod_{i=1,\cdots,h+1} \omega_{i1} \cdot \prod_{\substack{i=1,\cdots,h+1 \\ j=h+2,\cdots,n}} \omega_{ij} \cdot \sum_{l=1}^{h+1} \omega_{ll}\right] = 0. \tag{3.11}$$

This condition is only satisfied if

$$\omega_{22} + \omega_{33} + \cdots + \omega_{h+1,h+1} \equiv 0 \pmod{\omega_{11}} \tag{3.12}$$

and since (3.5) is the only one relation between the relative components, (3.12) holds only if $h = 0$ or $h + 1 = n$.

For $h = 0$ the density has the value $dS_0 = [\omega_{11} \cdots \omega_{1n}]$. If the points are assumed to be determined by their coordinates $a_1^1, a_1^2, \cdots, a_1^n$ taking into account (3.4) and (3.2) we get $dS_0 = [da_1^1 da_1^2 \cdots da_1^n]$ that is, the density for points equals the element of volume, which is an obvious result, since the center-affine transformations are assumed volume preserving.

For $h = n - 1$, if each hyperplane is determined by one, say a_1, of its points and the directions oa_i $(i = 2, \cdots, n)$ the density has the value

$$dS_{n-1} = [\omega_{11}\omega_{21} \cdots \omega_{n1}]. \tag{3.13}$$

In order to give a geometrical interpretation of this density we proceed as follows.

Let b be the point on the unit hypersphere of center o such that the radius ob is perpendicular to the hyperplane determined by the points o, a_i $(i = 2, 3, \cdots, n)$; that is, the point b determines the direction normal to S_{n-1}. If $b^1, b^2, \cdots, b^n$ are the coordinates of b, the element of area on the unit hypersphere corresponding to the point b is expressed by

$$dB = \frac{[db^1 \cdots db^{i-1}\, db^{i+1} \cdots db^n]}{(-1)^i b^i} \tag{3.14}$$

where the right side is independent of i. Let $b_2, b_3, \cdots, b_n$ be n-unit orthogonal vectors with the origin o on the hyperplane determined by $o, a_2, \cdots, a_n$. If we set

$$(b_i\, db) = -(b\, db_i) = \sum_{k=1}^{n} b_i^k\, db^k, \qquad i = 2, \cdots, n \tag{3.15}$$

will be

$$\left[\prod_{i=2}^{n} (b\, db_i)\right] = \sum_{i=1}^{n} \beta_i [db^1 \cdots db^{i-1}\, db^{i+1} \cdots db^n] \tag{3.16}$$

where β_i is the complementary minor of b^i in the determinant $| bb_2 \cdots b_n | = 1$. Therefore, taking into account (3.14) and (3.15) we get (in absolute value)

$$dB = \left[\prod_{i=2}^{n} (b\, db_i)\right]. \tag{3.17}$$

On the other hand, since the vectors b_i and oa_i are on the same hyperplane we may write

$$a_i = \sum_{k=2}^{n} \lambda_i^k b_k, \qquad i = 2, 3, \cdots, n \tag{3.18}$$

and consequently

$$\left[\prod_{i=2}^{n} (b\, da_i)\right] = |\lambda_i^k| \left[\prod_{k=2}^{n} (b\, db_k)\right] = |\lambda_i^k|\, dB. \tag{3.19}$$

Since $(ba_i) = 0$ for $i = 2, 3, \cdots, n$, from (3.3) we deduce $(bda_i) = \omega_{i1}(ba_1) = \omega_{i1}\rho_1 \cos\theta$, where ρ_1 is the length oa_1 and θ the angle which forms oa_1 with ob;

that is, if p means the distance from o to the hyperplane S_{n-1}, we have $p = \rho_1 \cos \theta$ and consequently from (3.19) we deduce

$$\left[\prod_{i=2}^{n} \omega_{i1}\right] p^{n-1} = |\lambda_i^k| \, dB. \tag{3.20}$$

Moreover, from (3.3) we deduce $(bda_1) = \omega_{11}(a_1b) = \omega_{11}p$ and since $(a_1b) = p$, we have $(bda_1) = dp - (a_1db)$; if, furthermore, we take into account $[(a_1db)dB] = 0$, we get

$$[\omega_{11}dB] \, p = [dpdB]. \tag{3.21}$$

On the other hand, since the parallelepiped spanned by $oa_i(i = 1, 2, \cdots n)$ has volume 1, it is $|\lambda_i^k| \, p = 1$, and therefore (3.13), (3.20) and (3.21) give

$$dS_{n-1} = p^{-(n+1)}[dpdB]. \tag{3.22}$$

This is the wanted geometrical interpretation for dS_{n-1}. We may summarize the foregoing results in the following

THEOREM 3.2. *The points and the hyperplanes are the only linear subspaces which have an invariant density with respect to the unimodular center-affine group $\mathfrak{A}$. The density for points equals the element of volume. The density for hyperplanes is given by* (3.22) *where p denotes the distance from the origin to the hyperplane and dB denotes the element of area on the unit n-dimensional sphere corresponding to the point which gives the direction normal to the hyperplane.*

For instance, if $p = p(B)$ is the support function with respect to the origin o, of a convex body in the n-dimensional space, which contains o, the measure of all hyperplanes exterior to the convex body, invariant with respect to $\mathfrak{A}$, is given by

$$M(S_{n-1}) = (1/n) \int p^{-n} \, dB \tag{3.23}$$

the integral extended over the whole surface of the n-dimensional sphere.

Finally, we want to give a geometrical interpretation for the cinematic density of $\mathfrak{A}$.

According to §1, the cinematic density of $\mathfrak{A}$ is given by

$$d\mathfrak{A} = \left[\prod_{i,k=1}^{n}{}' \omega_{ik}\right] \tag{3.24}$$

where the accent denotes that ω_{nn} is excluded.

From (3.4) and (3.2) we deduce

$$\left[\prod_{k=1}^{n} \omega_{ik}\right] = [da_i^1 \, da_i^2 \cdots da_i^n] = dA_i \tag{3.25}$$

where dA_i denotes the element of volume corresponding to the point a_i. According to (3.4) we have

$$\omega_{nk} = \alpha_k^1 da_n^1 + \alpha_k^2 da_n^2 + \cdots + \alpha_k^n da_n^n \tag{3.26}$$

where α_k^i denotes the algebraic complement of a_k^i in the determinant $| a_1 a_2 \cdots a_n | = 1$. Consequently

$$\left[\prod_{k=1}^{n-1} \omega_{nk}\right] = \sum_{j=1}^{n} \bar{\alpha}_n^j [da_n^1 \cdots da_n^{j-1}\, da_n^{j+1} \cdots da_n^n] \tag{3.27}$$

where $\bar{\alpha}_n^j$ denotes the algebraic complement of α_n^j in the adjoint determinant $| \alpha_i^j |$. By a known theorem (see, for instance, Kowalewski [5] p. 80) it is $\bar{\alpha}_n^j = (-1)^{n+j} a_n^j$. Consequently we have

$$\left[\prod_{k=1}^{n-1} \omega_{nk}\right] = \sum_{j=1}^{n} (-1)^{n+j}\, a_n^j [da_n^1 \cdots da_n^{j-1}\, da_n^{j+1} \cdots da_n^n]. \tag{3.28}$$

The right hand side of this expression is equal to n times the volume of the elementary cone, which projects from o the element of volume corresponding to the point a_n. If we represent it by dV_{a_n} from (3.24), (3.25) and (3.28) we get

$$d\mathfrak{A} = n[dA_1 dA_2 \cdots dA_{n-1} dV_{a_n}] \tag{3.29}$$

which is the geometrical interpretation for $d\mathfrak{A}$ we want to obtain.

This cinematic density can also be expressed in another form, which will be useful in the next section. It is based in the following

LEMMA 3.1. *Let $a_1, a_2, \cdots a_{n-1}$ be $n - 1$ points in the n-dimensional space; let dA_i be the element of volume corresponding to a_i and $d\bar{A}_i$ the element of $(n - 1)$-dimensional volume corresponding to a_i in the hyperplane S_{n-1} determined by the points $o, a_1, \cdots, a_{n-1}$. If dB denotes the element of area of the n-dimensional unit sphere corresponding to the point b such that the radius ob is normal to S_{n-1}, and $V(a_1 \cdots a_{n-1})$ represents the volume of the $(n - 1)$-dimensional parallelepiped spanned by the vectors oa_i $(i = 1, 2, \cdots, n - 1)$, then (in absolute value)*

$$[dA_1 dA_2 \cdots dA_{n-1}] = V(a_1 \cdots a_{n-1})[d\bar{A}_1 d\bar{A}_2 \cdots d\bar{A}_{n-1} dB]. \tag{3.30}$$

PROOF. Let ob_i $(i = 1, 2, \cdots, n - 1)$ be $n - 1$ orthogonal unit vectors contained in S_{n-1}, and w_{ik} $(k = 1, 2, \cdots, n - 1)$ be $n - 1$ orthogonal unit vectors orthogonal to a_i $(i = 1, 2, \cdots, n - 1)$ with $ow_{i,n-1} = ob$. If ρ_i is the length of oa_i, formula (3.17) applied to oa_i gives

$$dA_i = \left[\prod_{l=1}^{n-1} (w_{il}\, da_i)\, d\rho_i\right] \rho_i^{n-1} \tag{3.31}$$

and, by the same formula,

$$d\bar{A}_i = \left[\prod_{l=1}^{n-2} (w_{il}\, da_i)\, d\rho_i\right] \rho_i^{n-2}. \tag{3.32}$$

If it is,

$$a_i = \sum_{k=1}^{n-1} \lambda_i^k b_k, \qquad i = 1, 2, \cdots, n - 1 \tag{3.33}$$

will be

$$da_i = \sum_{k=1}^{n-1} d\lambda_i^k b_k + \sum_{k=1}^{n-1} \lambda_i^k \, db_k \tag{3.34}$$

and

$$(b \, da_i) = (w_{i,n-1} \, da_i) = \sum_{k=1}^{n-1} \lambda_i^k (b \, db_k). \tag{3.35}$$

Consequently, by exterior multiplication we get

$$\left[\prod_{i=1}^{n-1} (w_{i,n-1} \, da_i)\right] = |\lambda_i^k| \left[\prod_{k=1}^{n-1} (b \, db_k)\right] = |\lambda_i^k| \, dB. \tag{3.36}$$

From (3.31), (3.32) and (3.36) we deduce

$$[dA_1 \cdots dA_{n-1}] = |\lambda_i^k| \rho_1 \cdots \rho_{n-1} [d\bar{A}_1 \cdots d\bar{A}_{n-1} dB] \tag{3.37}$$

and since $|\lambda_i^k| \rho_1 \cdots \rho_{n-1} = V(a_1 \cdots a_{n-1})$, formula (3.30) is proved.

Let us now observe that if $dV_{a_{n-1}}$, analogously as in (3.29), means the volume of the elementary cone which projects from o the element of volume dA_{n-1} and $d\bar{V}_{a_{n-1}}$ has the analogue meaning in the subspace S_{n-1}, under the assumption $n > 2$, it is

$$dA_{n-1} = (n/\rho_{n-1})[dV_{a_{n-1}} d\rho_{n-1}], \qquad d\bar{A}_{n-1} = (n - 1/\rho_{n-1})[d\bar{V}_{a_{n-1}} d\rho_{n-1}]$$

and therefore, (3.30) may be written

$$n[dA_1 \cdots dA_{n-2} dV_{a_{n-1}}] = (n-1) V(a_1 \cdots a_{n-1}) [d\bar{A}_1 \cdots d\bar{A}_{n-2} d\bar{V}_{a_{n-1}} dB]. \tag{3.38}$$

By symmetry, the cinematic density (3.29) may also be written $d\mathfrak{A} = n\,[dA_1 \cdots dA_{n-2} dV_{a_{n-1}} dA_n]$ and therefore, from (3.38) we deduce

$$d\mathfrak{A} = (n-1) V(a_1 \cdots a_{n-1}) [d\bar{A}_1 \cdots d\bar{A}_{n-2} d\bar{V}_{a_{n-1}} dB dA_n]. \tag{3.39}$$

In order to introduce the cinematic density $d\mathfrak{A}_{n-1}$ of the unimodular center-affine group in the subspace S_{n-1}, it is enough to observe that by a change of variables $a_i^k = \rho a_i^{k*}$, $\rho = (V(a_1 \cdots a_{n-1}))^{1/n-1}$, it is

$$\begin{aligned} &(n-1)[d\bar{A}_1 \cdots d\bar{A}_{n-2} d\bar{V}_{a_{n-1}}] \\ &\qquad = (V(a_1 \cdots a_{n-1}))^{n-1} (n-1) [d\bar{A}_1^* \cdots d\bar{A}_{n-2}^* d\bar{V}_{a_{n-1}^*}]. \end{aligned} \tag{3.40}$$

In order to set in evidence the dimension of the space set now $d\mathfrak{A}_n$ instead of $d\mathfrak{A}$ and from (3.39) and (3.40) follows

$$d\mathfrak{A}_n = (V(a_1 \cdots a_{n-1}))^n \, [dA_n dB d\mathfrak{A}_{n-1}]. \tag{3.41}$$

If h is the distance from a_n to S_{n-1} it is $V(a_1 \cdots a_{n-1})h = 1$, and $d\mathfrak{A}_n$ takes the form

$$d\mathfrak{A}_n = h^{-n} [dA_n dB d\mathfrak{A}_{n-1}]. \tag{3.42}$$

(3.41) and (3.42) are two recurrent geometrical interpretations of $d\mathfrak{A}_n$ (for $n > 2$) which will be useful in the next section.

For the excluded case $n = 2$, (3.29) becomes in our actual notation

$$d\mathfrak{A}_2 = 2\rho_1^2[dA_2dB]. \tag{3.43}$$

4. A Theorem of Minkowski-Hlawka

In order to make an application of the results of the foregoing section we want to give a proof of the following theorem ammounced by Minkowski and first proved by Hlawka [4]:

If Q is an n-dimensional star domain of volume $< \zeta(n)$, then there exists a lattice of determinant 1 *such that Q does not contain any lattice point $\neq 0$.*

This theorem was also proved by C. L. Siegel [6] and A. Weil [8]. The following proof is based on the same idea as that of these authors. It is, however, more simple and exposed from a more elementary point of view.

Let us consider in the n-dimensional space S_n the lattice L_0 of points of entire coordinates, and the set of all lattices L transformed from L_0 by the group $\mathfrak{A}_n$, that is, the set of all lattices of modulo 1. In this case the subgroup Γ of $\mathfrak{A}_n$, which leaves invariant L_0, is discrete. In the space of parameters, Γ will be represented by a set of infinite isolated points. Consequently the invariant density for sets of lattices L will be the same (3.29) or (3.41) which now we will write

$$dL = (V(a_1 \cdots a_{n-1}))^n[dA_ndBd\mathfrak{A}_{n-1}] \tag{4.1}$$

under the assumption that L is determined by o and the points $a_1, \cdots a_n$.

In order to have a one-to-one correspondence between lattices L and points of $\mathfrak{A}_n$, we must consider not only the whole space $\mathfrak{A}_n$ but the space $\mathfrak{A}_n/\Gamma$. Consequently, in what follows, whenever dL appears under the integral sign, it must be understood that the integral is taken over $\mathfrak{A}_n/\Gamma$.

Let D be a given fixed domain of volume v in S_n and consider the integral

$$I = \int_{a_n \epsilon D} dL. \tag{4.2}$$

To evaluate this integral we first keep a_n fixed. For each given $\mathfrak{A}_{n-1}$, in order to obtain different lattices L, the points a_i $(i = 1, 2, \cdots, n - 1)$ can only vary in the intervals $a_i + \lambda a_n$ $(0 \leqq \lambda \leqq 1)$. Setting shortly $V(a_1 \cdots a_{n-1}) = V$ and considering the points Vb, we observe that they are contained in the hyperplane inverse of the hypersphere of diameter oa_n in the inversion of center o and power of inversion 1, because if h is the distance from a_n to the hyperplane determined by o and $a_1, a_2, \cdots, a_{n-1}$ it is always $Vh = 1$. Here, b has the same meaning as in the foregoing section.

Furthermore, when the points a_i describe the intervals $a_i + \lambda a_n (0 \leqq \lambda \leqq 1)$, independently of each other the point Vb describes the $(n - 1)$-dimensional parallelepiped spanned by the vectors $\alpha_i - \alpha_0$ $(i = 1, 2, \cdots, n - 1)$ where

$$\alpha_0 = \{a_1 \cdots a_{n-1}\} \text{ and}$$
$$\alpha_i = \{a_1 \cdots a_{i-1}a_i + a_n a_{i+1} \cdots a_{n-1}\}$$
$$= \{a_1 \cdots a_{n-1}\} + (-1)^{n-i}\{a_1 \cdots a_{i-1}a_{i+1} \cdots a_n\}.$$

The notation $\{a_1 a_2 \cdots a_{n-1}\}$ means the vector whose components are the determinants of order $n - 1$ in the rectangular matrix formed by the coordinates of $a_1, a_2, \cdots, a_{n-1}$.

The volume of the pyramid of vertex o and basis this parallelepiped has the value $(1/n) \mid \alpha_0 \cdots \alpha_{n-1} \mid = 1/n$, according to a well known property about adjoint determinants ([5] p. 80).

On the other hand, the volume of the last mentioned pyramid is given by $(1/n) \int V^n dB$. Consequently we get that the integral of $V^n dB$ in (4.2) has value 1.

If, only for a moment, we assume that the total measure of the unimodular center-affine transformations of the space S_{n-1} has a finite value V_{n-1}, from (4.1) we get $I = V_{n-1} \int_{a_n \in D} dA_n$. If now a_n describes D, we get the value $V_{n-1}v$. In this way each lattice L has been counted as many times as lattice points of coordinates primes among themselves (we shall say primitive lattice points) are contained in D. In fact, when a_n coincides with any one of these points it originates the same lattice. Consequently if we represent by N the number of primitive lattice points of L contained in D, we have the integral formula

$$\int N\, dL = vV_{n-1} \tag{4.3}$$

where the integration is extended over the whole space $\mathfrak{A}_n/\Gamma$.

In order to introduce in (4.3) instead of N, the total number $\bar{N}$ of lattice points contained in D for each lattice L, we follow a very useful device due to Siegel [6]. Let us consider the domain $i^{-1}D$ (of volume $i^{-n}v$), homothetic of D with respect to o and ratio i^{-1} (i integer). To every lattice point contained in D, whose coordinates have the greatest common diviser i (the number of which will be represented by N_i), corresponds a primitive lattice point in $i^{-1}D$. Therefore the same formula (4.3) applied to $i^{-1}D$, gives

$$\int N_i\, dL = vV_{n-1}i^{-n}, \qquad i = 2, 3, \cdots \tag{4.4}$$

Adding (4.3) and (4.4) for $i = 2, 3, \cdots$ we get

$$\int \bar{N}\, dL = vV_{n-1}\zeta(n). \tag{4.5}$$

It remains to evaluate V_{n-1}. For this purpose we may again follow the method of Siegel [6]. Let us consider the lattice of points of coordinates multiple of

m^{-1} (m integer); if $\bar{N}_m$ is the number of lattice points of this new lattice contained in D, (4.5) gives

$$\int (\bar{N}_m/m^n)\, dL = vV_{n-1}\zeta(n). \tag{4.6}$$

When $m \to \infty$, $\bar{N}_m/m^n$ tends to the volume v of D and therefore the total measure V_n of the lattices L will be

$$V_n = V_{n-1}\zeta(n). \tag{4.7}$$

Since formula (3.41) holds only for $n > 2$, (4.7) holds for $n = 3, 4, \cdots$. For the case $n = 2$, starting from (3.43) the same foregoing calculation gives $\int NdL = v$ instead of (4.3), $\int \bar{N}dL = v\zeta(2)$ instead of (4.5) and $V_2 = \zeta(2)$ instead of (4.7). Consequently (4.7) makes sure that every V_n is finite and gives, moreover, the known result $V_n = \zeta(2)\zeta(3) \cdots \zeta(n)$.

From (4.3), (4.5), and (4.7) we obtain the following mean values. If we consider all lattices L of modulo 1:

a) *The mean value of the number of primitive lattice points contained in a given domain D of volume v is $v/\zeta(n)$.*

b) *The mean value of the number of lattice points contained in D is equal to v.*

The announced theorem of Minkowski-Hlawka is an immediate consequence of a). In fact, if $v < \zeta(n)$, the mean value of primitive lattice points is < 1 and therefore there exist lattices without primitive lattice points in D. If D is a star domain, i.e., a point set which is measurable in the Jordan sense and which contains with any point x the whole segment λx, $0 \leqq \lambda \leqq 1$, and does not contain any primitive lattice point, it does not contain any lattice point and the theorem is proved.

5. The Unimodular Affine Group

The group $\mathfrak{R}$ of unimodular affine transformations, given by

$$(x^i)' = \sum_{j=1}^{n} a_j^i x^j + a_0^i, \qquad i = 1, 2, \cdots, n \tag{5.1}$$

with

$$| a_1 a_2 \cdots a_n | = 1 \tag{5.2}$$

may be studied in exactly the same way as in §2 and §3. Setting

$$da_i = \sum_{j=1}^{n} \omega_{ij} a_j, \qquad i = 0, 1, \cdots, n \tag{5.3}$$

it is found

$$\omega_{ij} = | a_1 \cdots a_{j-1}\, da_i\, a_{j+1} \cdots a_n |, \qquad \sum_{i=1}^{n} \omega_{ii} = 0 \tag{5.4}$$

with the equations of structure

$$\omega'_{ij} = \sum_{l=1}^{n} [\omega_{il}\omega_{lj}]. \tag{5.5}$$

Similarly, as in §2 and §3, we get in this case the following

THEOREM 5.1. *Except the points* ($h = 0$), *the linear subspaces have no invariant density with respect to the unimodular affine group. The sets* ($S_{h_1} + S_{h_2} + \cdots + S_{h_m}$), *with* $h_1 + h_2 + \cdots + h_m + m \leqq n + 1$, *of linear subspaces, admits a density only when either all* h_i *are equal to zero, or the condition*

$$h_1 + h_2 + \cdots + h_m + m - 1 = n$$

holds.

For instance, the sets of (hyperplane + point) admits a density, which is easily found to be equal to $p^{-(n+1)}dA_0\, dBdp$, where dA_0 is the element of volume corresponding to the point, dB is the element of area on the $(n - 1)$-dimensional unit sphere corresponding to the direction normal to the hyperplane, and p is the distance from the point to the hyperplane.

Finally, if the unimodular affine transformations are assumed determined by the point a_0 and n vectors $a_i - a_0$ ($i = 1, 2, \cdots, n$), the cinematic density is given by

$$d\mathfrak{K} = [dA_0 d\mathfrak{A}] \tag{5.6}$$

where dA_0 is the element of volume corresponding to a_0 and $d\mathfrak{A}$ is the cinematic density (3.29), (3.41) for the unimodular center-affine group with a_0 as fixed point.

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