INTEGRAL GEOMETRY OF THE PROJECTIVE GROUPS OF THE PLANE DEPENDING ON MORE THAN THREE PARAMETERS

BY

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Dedicated to Prof. Dr. Octav Mayer on the occasion of his seventieth birthday

1. *Introduction.* The groups of projectivities of the real projective plane on itself were given by Sophus Lie [3]. With the usual symbolism, those which depend on more than three parameters are the following (see G. Kowalewski [2], p. 324):

I. *Group depending on 8 parameters (general projective group)*

$$p,\ q,\ xp,\ xq,\ yp,\ yq,\ x(xp+yq),\ y(xp+yq).$$

There are not groups depending on 7 parameters.

II. *Groups depending on 6 parameters :*

1. $p,\ q,\ xp,\ yq,\ xq,\ x(xp+yq)$;
2. $p,\ q,\ xp,\ yq,\ xq,\ yp$.

III. *Groups depending on 5 parameters:*

1. $p,\ q,\ xq,\ 2xp+yq,\ x(xp+yq)$;
2. $p,\ q,\ xp,\ yq,\ xq$;
3. $p,\ q,\ yp,\ xq,\ xp-yq$.

IV. *Groups depending on 4 parameters:*

1. $p,\ q,\ xp,\ yq$;
2. $p,\ q,\ xq,\ xp+ayq$;
3. $p,\ q,\ xq,\ yq$;
4. $q,\ xp,\ xq,\ yq$;
5. $p,\ xp,\ yq,\ x(xp+yq)$.

In a previous paper [5], we solved the problem of finding all the groups of projectivities of the plane such that the sets of points or the sets of lines have an invariant measure with respect to them. In the present paper instead of sets of points or sets of lines, we consider sets of „pairs of elements" i.e. sets of pairs of points $(P_0 + P_1)$, pairs of lines $(G_0 + G_1)$, pairs of point and line $(P + G)$ and ask for the groups of projectivities (depending on more than three parameters in order that these pairs of elements may be transformed transitively), with respect to which the sets of such pairs of elements admit an invariant measure.

We need to recall the way of finding the relative components (forms of Maurer-Cartan) and the equations of structure for linear groups (see [6], [7]). We distinguish two cases:

1. *Groups of affine type.* In matrix notation they have the form

$$x' = Ax + B \tag{1.1}$$

where A is a non-singular matrix 2×2 and B a matrix 2×1; x and x' are the 2×1 matrices of the non-homogeneous coordinates of the point x and its image x'. The forms of Maurer-Cartan or relative components of the group are the elements of the matrices

$$\Omega_1 = A^{-1}\, dA\,, \quad \Omega_2 = A^{-1}\, dB \tag{1.2}$$

and the equations of structure are

$$d\Omega_1 = -\Omega_1 \wedge \Omega_1\,, \quad d\,\Omega_2 = -\Omega_1 \wedge \Omega_2 \tag{1.3}$$

where $\wedge$ denotes exterior product of differential forms.

2. *Groups of projective type.* They have the form

$$x' = Ax \tag{1.4}$$

where A is now a 3×3 matrix such that det. $A = 1$, and x, x' are the 3×1 matrices of the homogeneous coordinates of the point x and its transformed x'.

The forms of Maurer-Cartan are the elements of the matrix

$$\Omega = A^{-1}\, dA \tag{1.5}$$

and the equations of structure are, in matrix form,

$$d\Omega = -\Omega \wedge \Omega\,. \tag{1.6}$$

According to the general theory (see for instance [7]) if we have a set of relative components (or a set of linear combinations with constant coefficients of relative components), say $\omega_1, \omega_2, \dots, \omega_m$ such that the system $\omega_1 = 0$, $\omega_2 = 0, \dots, \omega_m = 0$ states the condition that a general pair of elements ($P+P$, $G+G$, $P+G$) be fixed in the plane, then the condition for the existence of an invariant density for such pairs of elements is that the exterior differential $d(\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_m)$ vanishes (having into account the equations of structure) and the invariant density (defined up to a constant factor) is then given by the exterior product $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_m$. The integral of this invariant density will be the invariant measure. In our case $m \leqslant 4$.

2. *The general real projective group of the plane.* The integral geometry of the real projective group of the plane

$$x' = \frac{ax+by+c}{mx+ny+r}, \qquad y' = \frac{ex+gy+h}{mx+ny+r} \tag{2.1}$$

has been well studied (Varga [9], Luccioni [4], Santaló [7]).

Let A_0, A_1, A_2, be three non-collinear points with its homogeneous coordinates normalized such that

$$|A_0 A_1 A_2| = 1 \tag{2.2}$$

where the left side denotes the determinant whose rows are the homogeneous coordinates of A_0, A_1, A_2. With respect to the frame (A_0, A_1, A_2) any point x can be expressed in the form

$$x = x_0 A_0 + x_1 A_1 + x_2 A_2 \tag{2.3}$$

and the forms of Maurer-Cartan are defined by the equations (see [7])

$$\begin{aligned} dA_0 &= \omega_1 A_0 + \omega_2 A_1 + \omega_3 A_2, \\ dA_1 &= \omega_4 A_0 + \omega_5 A_1 + \omega_6 A_2, \\ dA_2 &= \omega_7 A_0 + \omega_8 A_1 + \omega_9 A_2. \end{aligned} \tag{2.4}$$

From these equations it follows that

$$\begin{aligned} \omega_1 &= |dA_0 A_1 A_2|, & \omega_2 &= |A_0\, dA_0\, A_2|, & \omega_3 &= |A_0 A_1\, dA_0|, \\ \omega_4 &= |dA_1 A_1 A_2|, & \omega_5 &= |A_0\, dA_1\, A_2|, & \omega_6 &= |A_0 A_1\, dA_1|, \\ \omega_7 &= |dA_2 A_1 A_2|, & \omega_8 &= |A_0\, dA_2\, A_2|, & \omega_9 &= |A_0 A_1\, dA_2|. \end{aligned} \tag{2.5}$$

Differentiating (2.2) one gets

$$\omega_1 + \omega_5 + \omega_9 = 0. \tag{2.6}$$

The equations of structure, according to (1.6) are the elements of the matrix

$$d\Omega = -\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_4 & \omega_5 & \omega_6 \\ \omega_7 & \omega_8 & \omega_9 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_4 & \omega_5 & \omega_6 \\ \omega_7 & \omega_8 & \omega_9 \end{pmatrix}$$

which write

$$\begin{aligned} &d\omega_1 = -\omega_2 \wedge \omega_4 - \omega_3 \wedge \omega_7, \quad d\omega_2 = -\omega_1 \wedge \omega_2 - \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_8, \\ &d\omega_3 = -\omega_1 \wedge \omega_3 - \omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_9, \quad d\omega_4 = -\omega_4 \wedge \omega_1 - \omega_5 \wedge \omega_4 - \omega_6 \wedge \omega_7 \\ (2.7)\quad &d\omega_5 = -\omega_4 \wedge \omega_2 - \omega_6 \wedge \omega_8, \quad d\omega_6 = -\omega_4 \wedge \omega_3 - \omega_5 \wedge \omega_6 - \omega_6 \wedge \omega_9, \\ &d\omega_7 = -\omega_7 \wedge \omega_1 - \omega_8 \wedge \omega_4 - \omega_9 \wedge \omega_7, \quad d\omega_8 = -\omega_7 \wedge \omega_2 - \omega_8 \wedge \omega_5 - \omega_9 \wedge \omega_8, \\ &d\omega_9 = -\omega_7 \wedge \omega_3 - \omega_8 \wedge \omega_6 . \end{aligned}$$

It is well known that the sets of pairs of points and the sets of pairs of lines have not an invariant measure with respect to the projective group [7]. We now consider the following cases:

1. *Sets of point P and line G with $P \in G$*. If we take $P \equiv A_0$ and $G \equiv$ line $A_0 A_1$, in order that the pair $P + G$ be kept fixed under the group (2.1), according to (2.4) we have $\omega_2 = 0$, $\omega_3 = 0$, $\omega_6 = 0$. Using the equations of structure we get $d(\omega_2 \wedge \omega_3 \wedge \omega_6) = -4\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_6 - 2\omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6 \neq 0$. This shows that *the sets of points and line such that the point belongs to the line, have not an invariant measure with respect to the projective group.*

2. *Sets of point P and line G such that P is not on G*. If we take $P \equiv A_0$ and $G \equiv$ line $A_1 A_2$, in order that the pair $P + G$ be kept fixed we have $\omega_2 = 0$, $\omega_3 = 0$, $\omega_4 = 0$, $\omega_7 = 0$. Using the equations of structure we get $d(\omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_7) = 0$. Consequently we have: *the pairs $P + G$ with P not on G, have an invariant measure with respect to the projective group.*

This measure is the integral of the invariant density

$$d(P + G) = \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_7 . \tag{2.8}$$

We now want to obtain a geometrical interpretation of this density. Let $P \equiv A_0(a, b, c)$, $A_1(e, g, h)$, $A_2(m, n, r)$ be the homogeneous coordinates of the points P, A_1, A_2 normalized such that

$$\begin{pmatrix} a & b & c \\ e & g & h \\ m & n & r \end{pmatrix} = 1 . \tag{2.9}$$

From (2.4) we deduce

$$
\begin{gathered}
da = a\,\omega_1 + e\,\omega_2 + m\,\omega_3, \quad db = b\,\omega_1 + g\,\omega_2 + n\,\omega_3, \quad dc = c\,\omega_1 + h\,\omega_2 + r\,\omega_3, \\
de = a\,\omega_4 + e\,\omega_5 + m\,\omega_6, \quad dg = b\,\omega_4 + g\,\omega_5 + n\,\omega_6, \\
dh = c\,\omega_4 + h\,\omega_5 + r\,\omega_6, \\
dm = a\,\omega_7 + e\,\omega_8 + m\,\omega_9, \quad dn = b\,\omega_7 + g\,\omega_8 + n\,\omega_9, \quad dr = c\,\omega_7 + h\,\omega_8 + r\,\omega_9.
\end{gathered} \tag{2.10}
$$

The non-homogeneous coordinates of P are $x = a/c$, $y = b/c$. Therefore, calling $dP = dx \wedge dy$ = metric density for points on the plane (= element of area), we have

$$
dP = dx \wedge dy = \frac{1}{c^4}(c\,da - a\,dc) \wedge (c\,db - b\,dc)
$$

and by substituting the values (2.10) and having into account (2.9) we get

$$
dP = \frac{1}{c^3}\,\omega_2 \wedge \omega_3. \tag{2.11}
$$

The line $G \equiv A_1 A_2$ is

$$
y = -\frac{nh - gr}{er - mh}\,x - \frac{gm - en}{er - mh}. \tag{2.12}
$$

If we call (p, φ) the normal coordinates for lines on the plane (p = distance from the origin to the line, φ = angle between the normal to the line and the x-axis), we have

$$
\frac{nh - gr}{er - mh} = \cot\varphi, \quad -\frac{gm - en}{er - mh} = \frac{p}{\sin\varphi}.
$$

Differentiating these equations and using (2.10) we have

$$
\begin{gathered}
\frac{d\varphi}{\sin^2\varphi} = \frac{r\,\omega_4 - h\,\omega_7}{(er - mh)^2}, \\
\frac{dp}{\sin\varphi} + (\ldots)\,d\varphi = \frac{-m\,\omega_4 + e\,\omega_7}{(er - mh)^2}.
\end{gathered} \tag{2.13}
$$

By exterior multiplication of (2.13) and (2.11), calling $dG = dp \wedge d\varphi$ = metric density for lines on the plane (see [7]), we get

$$
\frac{dP \wedge dG}{\sin^3\varphi} = \frac{\omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_7}{c^3 (er - mh)^3} = \frac{d(P + G)}{c^3 (er - mh)^3}.
$$

The distance from $P(a/c, b/c)$ to the line (2.12) has the value

$$\delta = \frac{\sin\varphi}{c(er - mh)} .$$

Therefore we have

$$d(P + G) = \frac{dP \wedge dG}{\delta^3} \tag{2.14}$$

and we may state the following theorem:

The sets of pairs of point P and line G, such that P is not on G, have an invariant measure with respect to the general projective group. This measure is given by the integral of the invariant density (2.14), *where* δ *is the distance from P to G.*

c) *Sets of triples of non-collinear points.* In order that the points $P_0 \equiv A_0$, $P_1 \equiv A_1$ and $P_2 \equiv A_2$ be fixed under the projective group, according to (2.4) we have the conditions $\omega_2 = \omega_3 = 0$, $\omega_4 = \omega_6 = 0$, $\omega_7 = \omega_8 = 0$. An easy computation, having into account the equations of structure, gives $d(\omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8) = 0$. Consequently, the sets of triples of non-collinear points have an invariant density with respect to the projective group. This density has the value

$$d(P_0 + P_1 + P_2) = \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8 .$$

In order to obtain a geometric characterization of this density, we observe that the non homogeneous coordinates of P_0, P_1, P_2 are

$$P_0(a/c,\ b/c), \quad P_1(e/h,\ g/h), \quad P_2(m/r,\ n/r) .$$

We know that $dP_0 = dx_0 \wedge dy_0 = c^{-3}\omega_2 \wedge \omega_3$ (2.11). By direct computation we obtain, using (2.10).

$$dx_1 = d(e/h) = \frac{(ha - ec)\,\omega_4 + (hm - er)\,\omega_6}{h^2},$$

$$dy_1 = d(g/h) = \frac{(hb - gc)\,\omega_4 + (hn - gr)\,\omega_6}{h^2},$$

$$dP_1 = dx_1 \wedge dy_1 = -\,h^{-3}\,\omega_4 \wedge \omega_6 ,$$

and analogously

$$dP_2 = dx_2 \wedge dy_2 = -\,r^{-3}\,\omega_2 \wedge \omega_8 .$$

Therefore we have (up to the sign, which is not essential since we consider allways densities in absolute value)

$$dP_0 \wedge dP_1 \wedge dP_2 = \frac{d(P_0 + P_1 + P_2)}{c^3 h^3 r^3} ; \tag{2.15}$$

The area of the triangle P_0, P_1, P_2 is

$$T = \frac{1}{2}\begin{vmatrix} \frac{a}{c} & \frac{b}{c} & 1 \\ \frac{e}{h} & \frac{g}{h} & 1 \\ \frac{m}{r} & \frac{n}{r} & 1 \end{vmatrix} = \frac{1}{2\,chr}. \tag{2.16}$$

From (2.15) and (2.16) we deduce (up to a constant factor)

$$d(P_0 + P_1 + P_2) = \frac{dP_0 \wedge dP_1 \wedge dP_2}{T^3}. \tag{2.17}$$

Therefore we can state: *The sets of triples of non-collinear points (= sets of triangles) have an invariant measure with respect to the projective group, which is given by the integral of the density* (2.17).

This result was given by Stoka [8] and Luccioni [4].

If we want to introduce the density of the three lines G_0, G_1, G_2 which from the triangle $P_0 P_1 P_2$, according to a formula of Blaschke [1], p. 53, we have

$$d(G_0 + G_1 + G_2) = \left(\frac{D}{T}\right)^3 dG_0 \wedge dG_1 \wedge dG_2$$

where D is the diameter of the circle circumscribed to the triangle fromed by the three lines G_0, G_1, G_2.

3. *Projective groups depending on six parameters.* The projective groups depending on 6 parameters are the following:

1. $p, q, xp, yq, xq, x(xp + xq)$. The explicit equations are

$$x' = \frac{ax + by + c}{ny + r}, \quad y' = \frac{my + h}{ny + r}, \quad a(mr - nh) = 1. \tag{3.1}$$

Taking the three points $A_0(a, b, c)$, $A_1(0, m, h)$, $A_2(0, n, r)$ as the vertices of the coordinate system, the relative components are defined by the following equations (see [7])

$$\begin{aligned} dA_0 &= \omega_1 A_0 + \omega_2 A_1 + \omega_3 A_2, \\ dA_1 &= \qquad\quad\ \omega_4 A_1 + \omega_5 A_2, \\ dA_2 &= \qquad\quad\ \omega_6 A_1 + \omega_7 A_2, \end{aligned} \tag{3.2}$$

from which we deduce

$$\omega_1 = \frac{da}{a}, \quad \omega_2 = (nc - br)\,da + ar\,db - an\,dc,$$

$$\omega_3 = (bh - mc)\,da - ah\,db + am\,dc, \quad \omega_4 = ar\,dm - an\,dh,$$

$$\omega_5 = -ah\,dm + am\,dh, \quad \omega_6 = ar\,dn - an\,dr, \quad \omega_7 = -ah\,dn + am\,dr$$

with the condition

$$\omega_1 + \omega_4 + \omega_7 = 0.$$

From these equations we deduce

$$da = a\,\omega_1, \quad db = m\,\omega_2 + n\,\omega_3 + b\,\omega_1,$$

$$dc = r\,\omega_3 + h\,\omega_2 + c\,\omega_1, \quad dm = m\,\omega_4 + n\,\omega_5,$$

$$dh = r\,\omega_5 + h\,\omega_4, \quad dn = m\,\omega_6 + n\,\omega_7, \quad dr = h\,\omega_6 + r\,\omega_7.$$

The equations of structure are

$$d\omega_1 = 0, \quad d\omega_2 = -\omega_1 \wedge \omega_2 - \omega_2 \wedge \omega_4 - \omega_3 \wedge \omega_6,$$

$$d\omega_3 = -\omega_1 \wedge \omega_3 - \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_7, \quad d\omega_4 = -\omega_5 \wedge \omega_6,$$

$$d\omega_5 = -\omega_4 \wedge \omega_5 - \omega_5 \wedge \omega_7, \quad d\omega_6 = -\omega_6 \wedge \omega_4 - \omega_7 \wedge \omega_6, \quad d\omega_7 = -\omega_6 \wedge \omega_5.$$

We have now at our disposal all necessary elements in order to analyze one by one all the following possible cases:

a) *Sets of point P and line G, with P on G.* Taking $P \equiv A_0$ and $G \equiv A_0 A_1$ the differential equations of the element $P + G$ are $\omega_2 = 0$, $\omega_3 = 0$, $\omega_5 = 0$. Since $d(\omega_2 \wedge \omega_3 \wedge \omega_5) = 2\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 + 2\,\omega_2 \wedge \omega_3 \wedge \omega_7 \wedge \omega_5 \neq 0$, we have: *the sets of pairs $P + G$ with $P \in G$ have not an invariant measure with respect to the group* (3.1).

b) *Sets of pairs of points $P_0 + P_1$.* Let us take $P_0 \equiv A_0$ and $P_1 \equiv A_0 + A_1$. In order that P_0 be kept fixed under the group (3.1) we have the conditions $\omega_2 = 0, \omega_3 = 0$. For $A_0 + A_1$ we have $d(A_0 + A_1) = \omega_1 (A_0 + A_1) + (\omega_2 + \omega_4 - \omega_1) A_1 + (\omega_3 + \omega_5) A_2$, and $A_0 + A_1$ will be fixed if $\omega_2 + \omega_4 - \omega_1 = 0$, $\omega_3 + \omega_5 = 0$. Consequently the possible invariant density for the pairs $P_0 + P_1$ is $\omega_2 \wedge \omega_3 \wedge (\omega_4 - \omega_1) \wedge \omega_5$. Since $d(\omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 - \omega_2 \wedge \omega_3 \wedge \omega_1 \wedge \omega_5) \neq 0$ we have: *the pairs of points $P_0 + P_1$ have not an invariant measure under the group* (3.1).

c) *Sets of pairs of lines $G_0 + G_1$.* Taking the pair $G_0 \equiv A_0 A_1$, $G_1 \equiv A_0 A_2$ we get the system $\omega_2 = 0$, $\omega_3 = 0$, $\omega_5 = 0$, $\omega_6 = 0$. Since $d(\omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6) = 3\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6 \neq 0$, we have: *the sets of pairs of lines have not an invariant measure under the group* (3.1).

Since (3.1) is a subgroup of the projective group, the sets of pairs $P+G$, such that P is not on G, will have the same invariant density (2.14) with respect to the group (3.1). The same obviously holds for all the groups that we consider in what follows.

2. p, q, xp, xq, yp, yq. It is the general affine group whose finite equations are

$$x' = ax + by + c, \quad y' = mx + ny + r \tag{3.3}$$

with the condition $an - bm \neq 0$. Putting

$$A = \begin{pmatrix} a & b \\ m & n \end{pmatrix}, \quad A^{-1} = \frac{1}{\Delta}\begin{pmatrix} n & -b \\ -m & a \end{pmatrix}, \quad \Delta = an - bm$$

and applying (1.2) we have

$$\Omega_1 = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} \omega_5 \\ \omega_6 \end{pmatrix}$$

where

$$\begin{aligned}
&\omega_1 = \frac{1}{\Delta}(n\,da - b\,dm), && \omega_2 = \frac{1}{\Delta}(n\,db - b\,dn), \\
&\omega_3 = \frac{1}{\Delta}(-m\,da + a\,dm), && \omega_4 = \frac{1}{\Delta}(-m\,db + a\,dn), \\
&\omega_5 = \frac{1}{\Delta}(n\,dc - b\,dr), && \omega_5 = \frac{1}{\Delta}(-m\,dc + a\,dr).
\end{aligned} \tag{3.4}$$

From these equations we deduce:

$$\begin{aligned}
&da = a\omega_1 + b\omega_3, \quad db = a\omega_2 + b\omega_4, \quad dc = a\omega_5 + b\omega_6, \\
&dm = n\omega_3 + m\omega_1, \quad dn = n\omega_4 + m\omega_2, \quad dr = n\omega_6 + m\omega_5.
\end{aligned} \tag{3.5}$$

The equations of structure are (according to (1.3))

$$\begin{aligned}
&d\omega_1 = \omega_2 \wedge \omega_3, \quad d\omega_2 = -\omega_1 \wedge \omega_2 - \omega_2 \wedge \omega_4, \\
&d\omega_3 = -\omega_3 \wedge \omega_1 - \omega_4 \wedge \omega_3, \quad d\omega_4 = -\omega_3 \wedge \omega_2, \\
&d\omega_5 = -\omega_1 \wedge \omega_5 - \omega_2 \wedge \omega_6, \quad d\omega_6 = -\omega_3 \wedge \omega_5 - \omega_4 \wedge \omega_6.
\end{aligned}$$

Let us consider the following cases:

a) *Sets of point and line $P+G$, with $P \in G$.* Under the group (3.3) the point (0,0) is transformed into the general point $P(c, r)$ and the line

$y = 0$ into the general line G: $y = (m/a)x - mc/a + r$. The pair $P + G$ is defined by the equations $\omega_5 = 0$, $\omega_6 = 0$, $\omega_3 = 0$ and we have $d(\omega_5 \wedge \omega_6 \wedge \omega_3) = 2\,\omega_5 \wedge \omega_4 \wedge \omega_6 \wedge \omega_3 \neq 0$. Consequently we get: *the sets of pairs $P + G$, with P on G, do not admit an invariant measure under the group* (3.3).

b) *Sets of pairs of parallel lines.* The pairs of parallel lines transform transitively by the group (3.3); therefore we may ask for the existence of an invariant measure for sets of parallel lines. The line $x = 0$ goes into the general line $G_0 : y = (n/b)x - nc/b + r$ and the line $x = 1$ into the line $G_1 : y = (n/b)x - na/b - nc/b + m + r$. The system which determines the pair $G_0 + G_1$ is $\omega_2 = 0$, $\omega_5 = 0$, $\omega_1 = 0$. Since $d(\omega_1 \wedge \omega_2 \wedge \omega_5) = \omega_1 \wedge \omega_2 \wedge \omega_4 \wedge \omega_5 \neq 0$, we have: *the pairs of parallel lines do not possess an invariant measure under the general affine group* (3.3).

c) *Sets of pairs of points $P_0 + P_1$.* The points (0,0) and (1,0) are transformed by the group (3.3) into the general pair $P(c, r)$, $P(a+c, m+r)$. This pair will be fixed if $\omega_5 = 0$, $\omega_6 = 0$, $\omega_1 = 0$, $\omega_3 = 0$. Since $d(\omega_1 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6) = -2\,\omega_1 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \wedge \omega_6 \neq 0$, we have: *the sets of pairs of points do not admit an invariant measure under the group* (3.3).

d) *Sets of pairs of lines $G_0 + G_1$.* The lines $x = 0$ and $y = 0$ transform into the general pair of lines G_0: $y = (n/b)x - nc/b + r$, G_1: $y = (m/a)x - mc/a + r$ which correspond to the system $\omega_3 = 0$, $\omega_6 = 0$, $\omega_2 = 0$, $\omega_5 = 0$. Since $d(\omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6) \neq 0$ we have: *the pairs of lines have not an invariant measure with respect to the group* (3.3).

e) *Sets of parallelograms.* The parallelograms transform transitively under the group (3.3); since they depend on 6 parameters, the same that the group (3.3), it follows that the set of parallelograms have an invariant measure under the general affine group, which coincides with the measure of the group itself (cinematic measure in terms of the integral geometry). In order to have a geometrical interpretation of this measure, we observe that the square of vertices (0, 0) (0, 1), (1, 0) (1, 1) transforms into the general parallelogram Q of vertices $P_0(c, r)$, $P_1(b + c, n + r)$, $P_2(a + c, m + r)$, $P_3(a + b + c, m + n + r)$. According to (3.5) we have

$$dP_0 = \Delta\omega_5 \wedge \omega_6, \quad dP_1 = \Delta\omega_2 \wedge \omega_4, \quad dP_2 = \Delta\omega_1 \wedge \omega_3.$$

Therefore if we call $dQ = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \wedge \omega_6$ = cinematic density of the general affine group=density for parallelograms, we may write $dP_0 \wedge dP_1 \wedge dP_2 = \Delta^3 dQ$ and since $\Delta = S$ = area of the parallelogram Q, we have

$$dQ = \frac{dP_0 \wedge dP_1 \wedge dP_2}{S^3} \tag{3.6}$$

where P_0, P_1, P_2 are three vertices of the parallelogram Q and S is the area of Q. This expression (3.6) coincides with the density for sets of triangles (up to a constant factor) (2.17), as it should be, since each triangle determines one parallelogram and vice-versa.

Another expression for the density (3.6) was given by Stoka [8].

4. *Projective groups depending on five parameters.* The projective groups of the plane depending on five parameters are the following:

1. $p, q, xq, 2xp + yq, x(xp + yq)$. The finite equations of this group are

$$x' = \frac{ax + c}{mx + r}, \quad y' = \frac{bx + y + h}{mx + r}, \quad ar - mc = 1. \tag{4.1}$$

It is a group of projective type, which corresponds to the matrix

$$A = \begin{pmatrix} a & 0 & c \\ b & 1 & h \\ m & 0 & r \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} r & 0 & -c \\ -br + mh & 1 & -ah + bc \\ -m & 0 & a \end{pmatrix}$$

with the condition det. $A = 1$. The relative components are given by

$$\Omega = A^{-1}\,dA = \begin{pmatrix} \omega_1 & 0 & \omega_2 \\ \omega_3 & 0 & \omega_4 \\ \omega_5 & 0 & \omega_6 \end{pmatrix}$$

and have the values

$$\begin{aligned} \omega_1 &= r\,da - c\,dm, \quad \omega_2 = r\,dc - c\,dr, \\ \omega_3 &= (mh - br)\,da + db + (bc - ah)\,dm, \\ \omega_4 &= (mh - br)\,dc + dh + (bc - ah)\,dr, \\ \omega_5 &= -m\,da + a\,dm, \quad \omega_6 = -m\,dc + a\,dr \end{aligned}$$

with the condition

$$\omega_1 + \omega_6 = 0.$$

From these equations we deduce

$$\begin{aligned} &da = a\omega_1 + c\omega_5, \quad db = \omega_3 + h\omega_5 + b\omega_1, \quad dc = a\omega_2 - c\omega_1, \\ &dh = \omega_4 - h\omega_1 + b\omega_2, \quad dm = m\omega_1 + r\omega_5, \quad dr = m\omega_2 - r\omega_1. \end{aligned} \tag{4.2}$$

The equations of structure, according to (1.6) are

$$\begin{aligned} &d\omega_1 = -\,\omega_2 \wedge \omega_5, \quad d\omega_2 = -2\omega_1 \wedge \omega_2, \quad d\omega_3 = -\,\omega_3 \wedge \omega_1 - \omega_4 \wedge \omega_5, \\ &d\omega_4 = -\omega_3 \wedge \omega_2 - \omega_1 \wedge \omega_4, \quad d\omega_5 = -2\omega_5 \wedge \omega_1, \quad d\omega_6 = -\omega_5 \wedge \omega_2, \end{aligned}$$

We have now all necessary elements for analyzing all possible cases one by one.

a) *Sets of point and line $P+G$ such that P is on G.* The point (0,0) transforms into the general point $P(c/r, h/r)$ and the line $y=0$ into the line $G: y=(br-hm)x+ha-bc$. The equations which define the pair $P+G$ are $\omega_2=0$, $\omega_4=0$, $\omega_3=0$. We have $d(\omega_2\wedge\omega_3\wedge\omega_4)=$ $=-2\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4\neq 0$. Therefore: *the sets of pairs of point and line, such that the point is on the line, have not an invariant measure with respect to the group* (4.1).

b) *Sets of pairs of points P_0+P_1.* The points (0,0) and (1,0) transform into the general pair of points $P_0(c/r, h/r)$, $P_1((a+c)/(m+r),$ $(b+h)/(m+r))$. An easy calculation, having into account (4.2) gives

$$dP_0=dx_0\wedge dy_0=\frac{\omega_2\wedge\omega_4}{r^3},\quad dP_1=dx_1\wedge dy_1=\frac{(-\omega_5+2\omega_1+\omega_2)\wedge(\omega_3+\omega_4)}{(m+r)^3}$$

where x_0, y_0 are the coordinates of P_0 and x_1, y_1 those of P_1. Since $d(\omega_2\wedge\omega_4\wedge(-\omega_5+2\omega_1)\wedge\omega_3)=0$, it follows that the pairs of points have an invariant measure under the group (4.1), which is given by the integral of the following invariant density

$$d(P_0+P_1)=\omega_2\wedge\omega_4\wedge(2\omega_1-\omega_5)\wedge\omega_3=r^3(m+r)^3\,dP_0\wedge dP_1.$$

Having into account that

$$ar-mc=1,\quad a=(m+r)x_1-rx_1,\quad c=rx_0$$

we get

$$r(m+r)=\frac{1}{x_1-x_0}$$

and therefore we have

$$d(P_0+P_1)=\frac{dP_0\wedge dP_1}{(x_1-x_0)^3}. \tag{4.3}$$

We can state: *The sets of pairs of points have an invariant measure under the group* (4.1), *given by the integral of the invariant density* (4.3).

c) *Sets of pairs of lines G_0+G_1.* The line $y=0$ goes into G_0: $y=(br-hm)x+ha-bc$ and the line $y=x$ goes into $G_1: y=(br-$ $-hm+r)x+ha-bc-c$. The general pair of lines G_0, G_1 will be kept fixed if $d(br-hm)=0$, $d(ha-bc)=0$, $dr=0$, $dc=0$. These equations are equivalent to $\omega_1=0$, $\omega_2=0$, $\omega_3=0$, $\omega_4=0$ and since $d(\omega_1\wedge\omega_2\wedge\omega_3\wedge$ $\wedge\omega_4)=0$, it follows that the pairs of lines G_0+G_1 have an invariant density with respect to the group (4.1) given by

$$d(G_0+G_1)=\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4.$$

In order to get a geometrical interpretation of this measure, we introduce the normal coordinates $G_0(p_0, \varphi_0)$, $G_1(p_1, \varphi_1)$ of the lines. We have

$$br - hm = -\cot \varphi_0, \quad ha - bc = \frac{p}{\sin \varphi_0}.$$

Differentiation and exterior product of these equalitie give (up to the sign),

$$\omega_1 \wedge \omega_2 \frac{dG_0}{\sin^3 \varphi_0}, \text{ where } dG_0 = dp \wedge d\varphi_0.$$

Analogously we have

$$\omega_3 \wedge \omega_4 = \frac{dG_1}{\sin^3 \varphi_1}, \quad dG_1 = dp_1 \wedge d\varphi_1,$$

and consequently we have

$$d(G_0 + G_1) = \frac{dG_0 \wedge dG_1}{\sin^3 \varphi_0 \sin^3 \varphi_1}. \tag{4.4}$$

Thus we have established: *With respect to the group* (4.1), *the pairs of lines have the invariant density* (4.4).

2. p, q, xp, xq, yq. The finite equations of this group are

$$x' = ax + m, \quad y' = bx + cy + h. \tag{4.5}$$

With the notations of n. 1 we have

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1/a & 0 \\ -b/ac & b/c \end{pmatrix}, \quad B = \begin{pmatrix} m \\ h \end{pmatrix},$$

$$\Omega_1 = A^{-1} dA = \begin{pmatrix} \omega_1 & 0 \\ \omega_2 & \omega_3 \end{pmatrix}, \quad \Omega_2 = A^{-1} dB = \begin{pmatrix} \omega_4 \\ \omega_5 \end{pmatrix},$$

where

$$\omega_1 = \frac{da}{a}, \quad \omega_2 = -\frac{b}{ac} da + \frac{db}{c}, \quad \omega_3 = \frac{dc}{c},$$

$$\omega_4 = \frac{dm}{a}, \quad \omega_5 = -\frac{b}{ac} dm + \frac{dh}{c}.$$

From these equations we deduce

$$\begin{aligned} &da = a\omega_1, \quad db = c\omega_2 + b\omega_1, \quad dc = c\omega_3, \\ &dm = a\omega_4, \quad dh = c\omega_5 + b\omega_4. \end{aligned} \tag{4.6}$$

The equations of structure, according to (1.3) are

$$d\omega_1 = 0, \quad d\omega_2 = \omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_3, \quad d\omega_3 = 0,$$
$$d\omega_4 = -\omega_1 \wedge \omega_4, \; d\omega_5 = -\omega_2 \wedge \omega_4 - \omega_3 \wedge \omega_5.$$

We may now consider the following cases:

a) *Sets of pairs $P+G$ with $P \in G$.* Under the group (4.5) the point (0,0) is transformed into the general point $P(m,h)$ and the line $y=0$ into the $G: y=(b/a)x - bm/a + h$. In order that the pair $P+G$ be fixed we have the conditions $dm=0$, $dh=0$, $d(-bm/a)=0$ which are equivalent to $\omega_2=0$, $\omega_4=0$, $\omega_5=0$. We have $d(\omega_2 \wedge \omega_4 \wedge \omega_5) = 2\omega_2 \wedge \omega_3 \wedge \wedge \omega_4 \wedge \omega_5 \neq 0$ and therefore: *the sets of point P and line G such that P belongs to G, have not an invariant measure under the group* (4.5).

b) *Sets of parallel lines.* The lines $y=0$, $y=1$ transform into the general pair of parallel lines $G_0: y=(b/a)x - bm/a + h$, $G_1: y=(b/a)x - - bm/a + h + c$. These lines will be kept fixed under the group (4.5) if $\omega_2=0$, $\omega_3=0$, $\omega_5=0$. Since $d(\omega_2 \wedge \omega_3 \wedge \omega_5) = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \neq 0$ we have: *the pairs of parallel lines have not an invariant density under the group* (4.5).

c) *Sets of pairs of points P_0+P_1.* By the group (4.5) the point (0,0) goes into $P_0(m,h)$ and the point (1,0) into $P_1(a+m, b+h)$. These points will be fixed if $\omega_4=0, \omega_5=0, \omega_1=0, \omega_2=0$. Since $d(\omega_1 \wedge \omega_2 \wedge \omega_4 \wedge \omega_5) = = -2\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \neq 0$, we have: *the sets of pairs of points have not an invariant measure with respect to the group* (4.5).

d) *Sets of pairs of lines G_0+G_1.* By the group (4 5) the lines $y=0$, $y=x$ are transformed into the general pair of lines $G_0: y=(b/a)x - - bm/a + h$ and $G_1: y=((c+b)/a)x - m(c+b)/a + h$. The differential equations which define the first are $\omega_2=0$, $\omega_5=0$ and those wich define the second are $\omega_3-\omega_1=0$, $\omega_5-\omega_4=0$. Now we have $d(\omega_2 \wedge \omega_5 \wedge \wedge (\omega_3-\omega_1) \wedge \omega_4) = 2\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \neq 0$, and therefore: *the sets of pairs of lines G_0+G_1 have not an invariant measure with respect to the group* (4.5)

3. $p, q, yp, xq, xp-yq$. This is the affine unimodular group:

$$x' = ax + by + c, \quad y' = mx + gy + h, \quad ag - bm = 1. \tag{4.7}$$

With the notations of n. 1 we have

$$A = \begin{pmatrix} a & b \\ m & g \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} g & -b \\ -m & a \end{pmatrix}$$

$$\Omega_1 = A^{-1}dA = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}, \quad \Omega_2 = A^{-1}dB = \begin{pmatrix} \omega_5 \\ \omega_6 \end{pmatrix}$$

where

$$\omega_1 = g\,da - b\,dm, \quad \omega_2 = g\,db - b\,dg, \quad \omega_3 = -m\,da + a\,dm,$$

$$\omega_4 = -m\,db + a\,dg, \quad \omega_5 = g\,dc - b\,dh, \quad \omega_6 = -m\,dc + a\,dh,$$

with

$$\omega_1 + \omega_4 = 0.$$

From these equations we deduce

$$da = a\omega_1 + b\omega_3, \quad db = a\omega_2 + b\omega_4, \quad dc = a\omega_5 + b\omega_6,$$
$$dm = g\omega_3 + m\omega_1, \quad dg = g\omega_4 + m\omega_2, \quad dh = g\omega_6 + m\omega_5. \tag{4.8}$$

The equations of structure are

$$d\omega_1 = -\omega_2 \wedge \omega_3, \; d\omega_2 = -\omega_1 \wedge \omega_2 - \omega_2 \wedge \omega_4, \; d\omega_3 = 2\omega_1 \wedge \omega_3,$$

$$d\omega_4 = \omega_2 \wedge \omega_3, \; d\omega_5 = -\omega_1 \wedge \omega_5 - \omega_2 \wedge \omega_6, \; d\omega_6 = -\omega_3 \wedge \omega_5 - \omega_4 \wedge \omega_6.$$

Details on this group for the n-dimensional case were given in [6].

a) *Sets of point P and line G such that P is on G.* The point $(0,0)$ transforms into $P(c, h)$ and the line $y=0$ into the line $G: y = (m/a)x - cm/a + h$. The equation of this pair of elements are $\omega_5 = 0$, $\omega_6 = 0$, $\omega_3 = 0$ and since $d(\omega_3 \wedge \omega_5 \wedge \omega_6) = 2\omega_3 \wedge \omega_4 \wedge \omega_5 \wedge \omega_6 \neq 0$, we have: *the sets of pairs $P + G$ with $P \in G$ have not an invariant measure with respect to the group* (4.7).

b) *Sets of pairs of parallel lines.* It is known that the sets of parallel lines have an invariant measure under the unimodular affine group, which is given by the integral of the following invariant density (see [6])

$$d(G_0 + G_1) = \frac{dG_0 \wedge dp_1}{\Delta^4}$$

where Δ is the distance between the two parallel lines $G_0(p_0, \varphi_0)$ and $G_1(p_1, \varphi_0)$.

c) *Sets of pairs of points.* Since the unimodular affine group is area preserving, the pairs of points have obviously the invariant density

$$d(P_0 + P_1) = dP_0 \wedge dP_1.$$

d) *Sets of pairs of lines.* The group (4.7) carries the line $y = 0$ into $G_0: y = (m/a)x - cm/a + h$ and the line $x = 0$ into $G_1: y = (g/b)x - gc/b + h$. In order that these lines be fixed, we have

$$d\left(\frac{m}{a}\right) = \frac{\omega_3}{a^2} = 0, \quad d\left(\frac{g}{b}\right) = -\frac{\omega_2}{b^2} = 0,$$
$$d\left(-\frac{cm}{a} + h\right) = -\frac{c}{a^2}\omega_3 + \frac{1}{a}\omega_6 = 0, \; d\left(-\frac{gc}{b} + h\right) = \frac{c}{b^2}\omega_2 - \frac{1}{b}\omega_5 = 0. \tag{4.9}$$

The density for pairs of lines $G_0 + G_1$ will be

$$d(G_0 + G_1) = \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6$$

and it is effectively an invariant density since $d(\omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6) = 0$.

In order to obtain a geometrical interpretation of this density we introduce the normal coordinates p_0, φ_0 and p_1, φ_1 of G_0, G_1 and we have

$$(4.10) \quad \frac{m}{a} = -\cot \varphi_0, -\frac{cm}{a} + h = \frac{p_0}{\sin \varphi_0}, \quad \frac{g}{b} = -\cot \varphi_1, \quad -\frac{gc}{b} + h = \frac{p_1}{\sin \varphi_1}.$$

These equalities, together with (4.9), give

$$\frac{\omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6}{a^3 b^3} = \frac{dG_0 \wedge dG_1}{\sin^3 \varphi_0 \sin^3 \varphi_1}$$

where $dG_0 = dp_0 \wedge d\varphi_0$ and $dG_1 = dp_1 \wedge d\varphi_1$. From (4.10) we deduce $m = -a \cot \varphi_0$, $g = -b \cot \varphi_1$. Therefore we have

$$ag - bm = ab(\cot \varphi_0 - \cot \varphi_1) = 1$$

and consequently we can write

$$(4.11) \quad d(G_0 + G_1) = \frac{dG_0 \wedge dG_1}{\sin^3(\varphi_1 - \varphi_0)}.$$

Thus we can state: *the sets of pairs of lines $G_0 + G_1$ have an invariant density with respect to the unimodular affine group, which is given by* (4.11).

5. *Projective groups depending on four parameters.* The projective groups of the plane depending on four parameters are the following:

1. p, q, xp, yq. The finite equations of this group are

$$(5.1) \quad x' = ax + b, \quad y' = cy + h.$$

The relative components are

$$\omega_1 = \frac{da}{a}, \quad \omega_2 = \frac{dc}{c}, \quad \omega_3 = \frac{db}{a}, \quad \omega_4 = \frac{dh}{c}$$

and the equations of structure are

$$d\omega_1 = 0, \quad d\omega_2 = 0, \quad d\omega_3 = -\omega_1 \wedge \omega_3, \quad d\omega_4 = -\omega_2 \wedge \omega_4.$$

We are now in position to consider the following cases:

a) *Sets of point P and line G such that P is on G.* By the group (5.1) the point (0,0) goes into $P(b, h)$ and the line $y = x$ into the line $G: y = (c/a)x - cb/a + h$. The pair $P + G$ will be fixed if

$\omega_3 = 0$, $\omega_4 = 0$, $\omega_2 - \omega_1 = 0$. Since $d(\omega_3 \wedge \omega_4 \wedge (\omega_2 - \omega_1)) = -2\omega_1 \wedge \omega_2 \wedge \wedge \omega_3 \wedge \omega_4 \neq 0$ we have: *the pairs $P + G$ with P on G, have not an invariant measure with respect to the group* (5.1).

b) *Sets of pairs of parallel lines.* The lines $y = x$ and $y = x + 1$ are transformed into the general pair of parallel lines $G_0 : y = (c/a)\, x - cb/a + h$, $G_1 : y = (c/a)x - bc/a + h + c$. We have

$$d\left(\frac{c}{a}\right) = \frac{c}{a}(\omega_2 - \omega_1),$$

$$d\left(-\frac{cb}{a} + h\right) = -\frac{bc}{a}(\omega_2 - \omega_1) - c\omega_3 + c\omega_4, \tag{5.2}$$

$$d\left(-\frac{cb}{a} + c + h\right) = d\left(-\frac{cb}{a} + h\right) + c\omega_2.$$

The pair of parallel lines $G_0 + G_1$ will be fixed if $\omega_2 - \omega_1 = 0$, $\omega_4 - \omega_3 = 0$, $\omega_2 = 0$. Since $d(\omega_1 \wedge \omega_2 \wedge (\omega_3 - \omega_4)) = 0$ it follows that the pairs of parallel lines admite an invariant measure with respect to the group (5.1). This measure is the integral of the invariant density $d(G_0 + G_1) = \omega_1 \wedge \omega_2 \wedge (\omega_3 - \omega_4)$. In order to obtain a geometrical interpretation of this density we observe that

$$\frac{c}{a} = \cot \varphi, \quad -\frac{cb}{a} + h = \frac{p}{\sin \varphi}, \quad c = \frac{\Delta}{\sin \varphi} \tag{5.3}$$

where p, φ are the normal coordinates of G_0 and Δ is the distance between G_0 and G_1. From (5.2) and (5.3) we deduce (up to the sign)

$$\frac{c^3}{a}\omega_1 \wedge \omega_2 \wedge (\omega_3 - \omega_4) = \frac{dG_0 \wedge dp_1}{\sin^4 \varphi}.$$

Hence

$$d(G_0 + G_1) = \frac{dG_0 \wedge dp_1}{\Delta^2 \sin \varphi \cos \varphi}. \tag{5.4}$$

We have got: *the sets of pairs of parallel lines have an invariant measure with respect to the group* (5.1); *this measure is given by the integral of the invariant density* (5.4).

c) *Sets of points $P_0 + P_1$.* The pairs of points transform transitively under the group (5.4). Therefore they possess an invariant density which coincides with the cinematic density of the group, i. e.

$$d(P_0 + P_1) = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4.$$

In order to have a geometrical interpretation of this density we observe that the point (1,0) goes into $P_0(a+b,h)$ and the point (0,1) into $P(b,c+h)$. Putting

$$a+b=x_0, \quad h=y_0, \quad b=x_1, \quad c+h=y_1$$

we have

$$\omega_1=\frac{dx_0}{a}-\frac{dx_1}{a}, \quad \omega_2=\frac{dy_1}{c}-\frac{dy_0}{c}, \quad \omega_3=\frac{dx_1}{a}, \quad \omega_4=\frac{dy_0}{c}$$

and therefore

$$d(P_0+P_1)=\frac{dP_0\wedge dP_1}{(x_1-x_0)^2(y_1-y_0)^2} \tag{5.5}$$

where we have substituted $a=x_0-x_1$, $c=y_1-y_0$.

The pairs of lines are not transitive under the group (5.1); therefore is meaningless to ask for an invariant measure.

A remark. Notice that by the substitution $x\to x+iy$, $y\to x-iy$ and setting

$$a=\varrho e^{i\alpha}, \quad c=\varrho e^{-i\alpha}, \quad b=A+iB, \quad h=A-iB$$

the group (5.1) goes into

$$x'=\varrho(x\cos\alpha-y\sin\alpha)+A, \quad y'=\varrho(x\sin\alpha+y\cos\alpha)+B \tag{5.6}$$

which is the *group of similitudes* of the plane. The density (5.4) writes now (up to a constant factor)

$$d(P_0+P_1)=\frac{dP_0\wedge dP_1}{\delta^4} \tag{5.7}$$

where δ is the distance P_0P_1.

The density (5.4) for sets of parallel lines takes now, with respect to the group (5.6) of similitudes, the more simple form

$$d(G_0+G_1)=\frac{dG_0\wedge dp_1}{\Delta^2}. \tag{5.8}$$

The formula (5.5) was given by Stoka [8].

2. $p, q, xq, xp+\alpha yq$. The finite equations of this group are

$$x'=ax+b, \quad y'=cx+a^{\alpha}y+h \tag{5.9}$$

where α is a fixed constant and a, b, c, h are the four parameters of the group.

The general method of n. 1 gives the following forms of Maurer-Cartan

$$\omega_1 = \frac{da}{a}, \quad \omega_2 = -\frac{c\,da}{a^{\alpha+1}} + \frac{dc}{a^\alpha}, \quad \omega_3 = \frac{db}{a}, \quad \omega_4 = -\frac{c\,db}{a^{\alpha+1}} + \frac{dh}{a^\alpha}$$

with the following equations of structure

$$d\omega_1 = 0,\ d\omega_2 = (1-\alpha)\omega_1 \wedge \omega_2,\ d\omega_3 = -\omega_1 \wedge \omega_3,\ d\omega_4 = -\omega_2 \wedge \omega_3 - \alpha(\omega_1 \wedge \omega_4).$$

a) *Sets of point P and line G with $P \in G$.* By the group (5.9) the point (0,0) goes into $P(b, h)$ and the line $y = 0$ into $G: y = (c/a)x - bc/a + h$. These elements will be fixed if

$$\omega_3 = 0,\ \omega_4 = 0,\ \omega_2 = 0.\ \text{Now,}\ d(\omega_2 \wedge \omega_3 \wedge \omega_4) = -2\alpha\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$$

and so we have: the pairs $P + G$ with $P \in G$ have an invariant measure under the group (5.9) only in the case $\alpha = 0$. In this case, putting

$$b = x_0, \quad h = y_0, \quad \frac{c}{a} = -\cot\varphi$$

where φ means the angle between the normal to G and the x-axis, we have

$$dx_0 = a\omega_3, \quad dy_0 = \omega_4 + (\ldots)\omega_3, \quad \frac{d\varphi}{\sin^2\varphi} = \frac{1}{a}\omega_2$$

and therefore

$$d(P + G) = \frac{dP \wedge d\varphi}{\sin^2\varphi}. \tag{5.10}$$

We have thus established:

The sets of pairs $P + G$, with P on G, have not an invariant measure under the group (5.9), *excepting the case $\alpha = 0$. In this case, such a measure exists and is given by the integral of the invariant density* (5.10).

b) *Sets of pairs of parallel lines.* The parallel lines $y = 0$ and $y = 1$ transform by the group (5.9) into G_0: $y = (c/a)x - cb/a + h$, and $G_1 : y = (c/a)x - cb/a + h + a^\alpha$ respectively. Therefore, the pairs of parallel lines transform transitively by the group (5.9) excepting for the case $\alpha = 0$. Let us assume $\alpha \neq 0$. We have

$$d\left(\frac{c}{a}\right) = a^{\alpha-1}\,.\,\omega_2, \quad d\left(-\frac{cb}{a} + h\right) = -b\,a^{\alpha-1}\omega_2 + a^\alpha\omega_4, \tag{5.11}$$

$$d\left(-\frac{cb}{a} + h + a^\alpha\right) = -ba^{\alpha-1}\omega_2 + a^\alpha\omega_4 + \alpha\,a^\alpha\omega_1.$$

The conditions for keeping G_0, G_1 fixed, are then $\omega_2 = 0$, $\omega_4 = 0$, $\omega_1 = 0$. Since $d(\omega_1 \wedge \omega_2 \wedge \omega_4) = 0$ we have that the sets of pairs of parallel lines have an invariant density with respect to the group (5.9), namely $d(G_0 + G_1) = \omega_1 \wedge \omega_2 \wedge \omega_4$.

In order to get a geometrical interpretation of this density we put

$$(5.12) \qquad \frac{c}{a} = -\cot \varphi\,, \quad -\frac{cb}{a} + h = \frac{p_0}{\sin\varphi}\,, \quad a^{\alpha} = \frac{\Delta}{\sin\varphi}$$

where (p_0, φ) are the normal coordinates of G_0 and Δ the distance between G_0 and G_1. From (5.11) and (5.12) we deduce (up to a constant factor)

$$(5.13) \qquad d(G_0 + G_1) = \frac{dG_0 \wedge dp_1}{\Delta^{\frac{3\alpha-1}{\alpha}} \sin^{\frac{\alpha+1}{\alpha}} \varphi}\,.$$

The following theorem summarizes our results:

The sets of pairs of parallel lines admit an invariant measure with respect to the group (5.9), *assuming* $\alpha \neq 0$. *This measure is given by the integral of the invariant density* (5.13).

c. *Sets of pairs of points and sets of pairs of lines.* The pairs of points transform transitively under the group (5.9); therefore the sets of pairs of points will have an invariant measure, which coincides with the cinematic measure of the group, i. e.

$$d(P_0 + P_1) = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4\,.$$

A geometrical interpretation of this measure may be obtained as follows. The pair of points (0,0), (1,0) transforms into the general pair of points $P_0(b, h)$, $P_1(a + b, c + h)$. Since

$$db = a\,\omega_3, \quad dh = a^{\alpha}\,\omega_4 + c\,\omega_3, \quad da = a\,\omega_1, \quad dc = a^{\alpha}\,\omega_2 + c\,\omega_1$$

we have

$$d(P_0 + P_1) = a^{\alpha+2}\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4\,.$$

If we call x_0, x_1 the abscissae of P_0 and P_1, observing that $a = x_1 - x_0$, we get the expression

$$(5.14) \qquad d(P_0 + P_1) = \frac{dP_0 \wedge dP_1}{(x_1 - x_0)^{\alpha+2}}\,.$$

For the case of sets of pairs of lines, we observe that the pair $y = 0, y = x$ transforms into the pair $G_0 : y = (c/a)\,x - cb/a + h$, $G_1 : y = (c/a + a^{\alpha-1})\,x - (c/a + a^{\alpha-1})\,b + h$. Thus, excepting the case $\alpha = 1$, the pairs of lines transform transitively, and therefore they possess the invariant measure $d(G_0 + G_1) = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$. In order to obtain a geometrical interpretation of this density, comparing the equations above of G_0, G_1 with their normal form we have

$$\frac{c}{a} = -\cot \varphi_0, \qquad -\frac{cb}{a} + h = \frac{p_0}{\sin \varphi_0}, \qquad \frac{c}{a} + a^{\alpha-1} = -\cot \varphi_1,$$

$$-\frac{cb}{a} + h - a^{\alpha-1}\, b = \frac{p_1}{\sin \varphi_1}.$$

From these equations and the expressions of the relative components of the group, we deduce

$$\frac{dG_0 \wedge dG_1}{\sin^3 \varphi_0 \sin^3 \varphi_1} = (\alpha - 1)\, a^{4\alpha-2}\, \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$$

and consequently, up to a constant factor,

$$d(G_0 + G_1) = \frac{(\sin \varphi_0 \sin \varphi_1)^{\frac{\alpha+1}{\alpha-1}}\, dG_0 \wedge dG_1}{(\sin(\varphi_1 - \varphi_0))^{\frac{4\alpha-2}{\alpha-1}}}. \tag{5.15}$$

In conclusion:

The sets of pairs of points and the sets of pairs of lines have respectively the invariant densities (5.14) *and* (5.15) *with respect to the group* (5.9). *For the case of lines, the group corresponding* $\alpha = 1$ *must be excluded.*

3. p, q, xq, yq. The finite equations of this group are

$$x' = x + a, \quad y' = bx + cy + h. \tag{5.16}$$

The forms of Maurer-Cartan are

$$\omega_1 = \frac{db}{c}, \quad \omega_2 = \frac{dc}{c}, \quad \omega_3 = -\frac{b\, da}{c} + \frac{dh}{c}, \quad \omega_4 = da \tag{5.17}$$

and the equations of structure

$$d\omega_1 = \omega_1 \wedge \omega_2, \quad d\omega_2 = 0, \quad d\omega_3 = -\omega_1 \wedge \omega_4 - \omega_2 \wedge \omega_3, \quad d\omega_4 = 0$$

as is easily deduced from the general method of n. 1.

a) *Sets of pairs of point P and line G such that P is on G.* The point (0,0) transforms into $P(a,h)$ and the line $y = 0$ into the line G: $y = bx - ba + h$. These elements will be fixed if $\omega_3 = 0, \omega_4 = 0, \omega_1 = 0$. Since $d(\omega_1 \wedge \omega_3 \wedge \omega_4) = 2\, \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \neq 0$, we deduce: *the sets of pairs* $P + G$ *with* $P \in G$ *do not possess an invariant measure under the group* (5.16).

b) *Sets of parallel lines.* The parallel lines $y = 0, y = 1$ transform into the general pair of parallel lines G_0: $y = bx - ba + h$, G_1: $y = bx - ba + h + c$ by the group (5.16). They will rest fixed if $\omega_1 = 0$, $\omega_3 = 0, \omega_2 = 0$. Since $d(\omega_1 \wedge \omega_2 \wedge \omega_3) = 0$, we have that the pairs $G_0 + G_1$

of parallel lines have the invariant density $d(G_0+G_1)=\omega_1\wedge\omega_2\wedge\omega_3$. In order to obtain a geometrical interpretation of this density we introduce the normal coordinates (p_0,φ_0) of G_0 and the distance $\Delta=p_1-p_0$ between both parallel lines. We have

$$b=-\cot\varphi_0,\qquad -ba+h=\frac{p_0}{\sin\varphi_0},\qquad c=\frac{\Delta}{\sin\varphi_0}.$$

Taking differentials and comparing with (5.17) we get

$$c\,\omega_1=\frac{d\varphi_0}{\sin^2\varphi_0},\qquad -ac\omega_1+c\omega_3=\frac{dp_0}{\sin\varphi_0}+(\dots)\,d\varphi_0,\qquad c\omega_2=\frac{dp_1}{\sin\varphi_0}+\dots$$

Exterior multiplication gives (up to the sign)

$$c^3\,\omega_1\wedge\omega_2\wedge\omega_3=\frac{dG_0\wedge dp_1}{\sin^4\varphi_0}.$$

which can be written

$$d(G_0+G_1)=\frac{dG_0\wedge dp_1}{\Delta^3\sin\varphi_0}. \tag{5.18}$$

Our result is:

The sets of parallel lines have an invariant measure under the group (5.16) *which is given by the integral of the invariant density* (5.18).

c) *Sets of pairs of points and sets of pairs of lines.* The pairs of points do not transform transitively under the group (5.16).

For the sets of pairs of lines we observe that the line $y=0$ goes into $G_0: y=bx-ba+h$ and the line $y=x$ into $G_1: y=(b+c)x-ba-ac+h$. If we introduce the normal coordinates, we have

$$b=-\cot\varphi_0,\qquad -ba+h=\frac{p_0}{\sin\varphi_0},\qquad b+c=-\cot\varphi_1,$$

$$-ba-ac+h=\frac{p_1}{\sin\varphi_1}.$$

Substituting in (5.17) we get

$$\omega_1=\frac{d\varphi_0}{c\sin^2\varphi_0},\qquad \omega_2=\frac{d\varphi_1}{c\sin^2\varphi_1}+A\,d\varphi_0,$$

$$\omega_3=\frac{dp_0}{c\sin\varphi_0}+B\,d\varphi_0,\qquad \omega_4=\frac{dp_1}{c\sin\varphi_1}+C\,d\varphi_0+D\,d\varphi_1+E\,dp_0$$

where A,B,C,D,E are expressions whose explicit form is not necessary. Thus

$$d(G_0+G_1)=\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4=\frac{\sin\varphi_0\sin\varphi_1\, dG_0\wedge dG_1}{\sin^4(\varphi_1-\varphi_0)};$$

this is *the invariant density for pairs of lines under the group* (5.16).

4. q, xp, xq, yq. The finite equations of the group are

$$x'=ax, \qquad y'=bx+cy+h. \tag{5.19}$$

We easily obtain

$$\omega_1=\frac{da}{a}, \quad \omega_2=-\frac{b\,da}{ac}+\frac{db}{c}, \quad \omega_3=\frac{dc}{c}, \quad \omega_4=\frac{dh}{c} \tag{5.20}$$

with the equations of structure

$$d\omega_1=0, \quad d\omega_2=\omega_1\wedge\omega_2+\omega_2\wedge\omega_3, \quad d\omega_3=0, \quad d\omega_4=-\omega_3\wedge\omega_4.$$

We shall consider the following cases:

a) *Pairs of point P and line G with P on G.* The point (1,0) goes into $P(a, b+h)$ and the line $y+x=1$ goes into $G: y=((b-c)/a)x+h+c$. These elements will be kept fixed if $\omega_1=0, \omega_2+\omega_4=0, \omega_3+\omega_4=0$. Since $d(\omega_1\wedge\omega_2\wedge\omega_3+\omega_1\wedge\omega_2\wedge\omega_4+\omega_1\wedge\omega_4\wedge\omega_3)\neq 0$ it follows that *the pairs $P+G$ with $P\in G$ do not possess an invariant density under the group* (5.19).

b) *Sets of parallel lines.* The group (5.19) maps the parallel lines $y=0, y=1$ into $y=(b/a)x+h$, $y=(b/a)x+h+c$. These lines will be fixed when $\omega_2=0, \omega_3=0, \omega_4=0$. Since $d(\omega_2\wedge\omega_3\wedge\omega_4)=\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4\neq 0$, it follows that *the sets of parallel lines do not admit an invariant measure with respect to the group* (5.19).

c) *Sets of pairs of points and sets of pairs of lines.* The sets of points do not transform transitively under the group (5.19).

For sets of pairs of lines we may take the lines $y=0, x+y=1$ which transform into the general pair $G_0: y=(b/a)x+h$, $y=((b-c)/a)x+h+c$. Introducing the normal coordinates of the lines we have

$$\frac{b}{a}=-\cot\varphi_0, \quad h=\frac{p_0}{\sin\varphi_0}, \quad \frac{b-c}{a}=-\cot\varphi_1, \quad h+c=\frac{p_1}{\sin\varphi_1}. \tag{5.21}$$

Differentiating an applying (5.20) we get

$$\omega_2=\frac{a\,d\varphi_0}{c\sin^2\varphi_0}, \quad \omega_4=\frac{dp_0}{c\sin\varphi_0}+A\,d\varphi_0, \quad \omega_1-\omega_3=\frac{a\,d\varphi_1}{c\sin^2\varphi_1}+B\,d\varphi_0,$$

$$\omega_3+\omega_4=\frac{dp_1}{c\sin\varphi_1}+B\,d\varphi_1.$$

Exterior multiplication gives (up to the sign)

$$d(G_0+G_1)=\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4=\frac{a^2\,dG_0\wedge dG_1}{c^4\sin^3\varphi_0\sin^3\varphi_1}.$$

From (5.21) we have

$$\frac{c}{a}=\cot\varphi_1-\cot\varphi_0,\qquad c=\frac{p_1}{\sin\varphi_1}-\frac{p_0}{\sin\varphi_0}$$

and therefore

$$d(G_0+G_1)=\frac{\sin\varphi_0\sin\varphi_1\,dG_0\wedge dG_1}{(p_1\sin\varphi_0-p_0\sin\varphi_1)^2\sin^2(\varphi_1-\varphi_0)}.$$

This is *the expression of the density for pairs of lines which is invariant under the group* (5.19).

5. $p, xp, yq, x(xp+yq)$. The finite equations of the group are

$$x'=\frac{ax+b}{cx+h},\quad y'=\frac{ey}{cx+h}. \tag{5.22}$$

This is a group of projective type; it corresponds to the matrix

$$A=\begin{pmatrix} a & 0 & b\\ 0 & e & 0\\ c & 0 & h\end{pmatrix},\quad e(ah-bc)=1. \tag{5.23}$$

The forms of Maurer-Cartan are the elements of the matrix

$$\Omega=A^{-1}dA=\begin{pmatrix}\omega_1 & 0 & \omega_2\\ 0 & \omega_3 & 0\\ \omega_4 & 0 & \omega_5\end{pmatrix}$$

and write

$$\begin{aligned} &\omega_1=eh\,da-be\,dc, \qquad \omega_2=eh\,db-be\,dh, \qquad \omega_3=\frac{de}{e},\\ &\omega_4=-ce\,da+ae\,dc, \qquad \omega_5=-ce\,db+ae\,dh. \end{aligned} \tag{5.24}$$

They satisfy the relation

$$\omega_1+\omega_3+\omega_5=0.$$

The equations of structure in matrix form are $d\Omega=-\Omega\wedge\Omega$ and in explicit form write

$$d\omega_1=-\omega_2\wedge\omega_4,\qquad d\omega_2=-2\,\omega_1\wedge\omega_2+\omega_2\wedge\omega_3,\qquad d\omega_3=0.$$
$$d\omega_4=2\,\omega_1\wedge\omega_4+\omega_3\wedge\omega_4,\qquad d\omega_5=\omega_2\wedge\omega_4.$$

From (5.24) we deduce

$$
\begin{aligned}
(5.25) \qquad da = a\,\omega_1 + b\,\omega_4,\quad db = a\,\omega_2 + b\,\omega_5,\quad dc = h\,\omega_4 + c\,\omega_1, \\
de = e\,\omega_3,\quad dh = h\,\omega_5 + c\,\omega_2.
\end{aligned}
$$

We have all necessary elements for considering the following cases:

a) *Sets of point P and line G such that P is on G.* The point (0,1) and the line $x + y = 1$ are maped under the group (5.22) into the general pair $P(b/h, e/h), G: y = -e^2(c+h)x + e^2(a+b)$. Making use of (5.25) we have that the pair $P + G$ will be kept fixed if $\omega_2 = 0, \omega_1 + 2\omega_3 = 0, 3\omega_3 + \omega_4 = 0$. Since $d(\omega_2 \wedge (\omega_1 + 2\omega_3) \wedge (3\omega_3 + \omega_4)) = 0$, we have that the sets of pairs $P + G$ with $P \in G$ have an invariant measure under the group (5.22). This measure is the integral of the invariant density

$$d(P + G) = \omega_2 \wedge (\omega_1 + 2\omega_3) \wedge (3\omega_3 + \omega_4).$$

In order to have the geometric meaning of this density we introduce the coordinates $x = b/h, y = e/h$ of P and the normal coordinates p, φ of G, given by the expressions

$$e^2(c + h) = \cot\varphi,\quad e^2(a+b) = \frac{p}{\sin\varphi}.$$

We have

$$dx = \frac{\omega_2}{eh^2},\quad dy = \frac{2\,eh\,\omega_3 + eh\,\omega_1 - ec\,\omega_2}{h^2}$$

$$\frac{d\varphi}{\sin^2\varphi}\, e^2(h - c)\,\omega_1 - e^2 c\,\omega_2 - e^2(2c + h)\,\omega_3 - e^2 h\,\omega_4.$$

Exterior multiplication gives (in absolute value)

$$\frac{dP \wedge d\varphi}{\sin^2\varphi} = \frac{e^2}{h^2}\, d(P + G)$$

and since $e/h = y$,

$$(5.26) \qquad d(P + G) = \frac{dP \wedge d\varphi}{y^2 \sin^2\varphi}.$$

Thus we have shown:

The sets of pairs $P + G$ with $P \in G$ have an invariant measure under the group (5.22); *this measure is the integral of the invariant density* (5.26).

It is worth while to mention that the group (5.22) and the group (5.9) for $\alpha = 0$ are the only projective groups depending on more than three parameters that have an invariant measure for the sets $P + G$ with $P \in G$.

b) *Sets of pairs of points and sets of pairs of lines*. Since the pairs of points $P_0 + P_1$ and the pairs of lines $G_0 + G_1$ transform transitively under the group (5.22) they will have an invariant measure, which coincides with the cinematic measure of the group, i. e. is the integral of the density $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$.

In order to get a geometrical interpretation of this density in both cases, we may proceed as follows.

The points (0,1) and (1,1) are carried by the group (5.22) into the general pair $P_0(b/h, e/h)$ and $P_1((a+b)/(c+h), e/(c+h))$. Setting

$$x_0 = \frac{b}{h}, \quad y_0 = \frac{e}{h}, \qquad x_1 = \frac{a+b}{c+h}, \quad y_1 = \frac{e}{c+h} \tag{5.27}$$

and using (5.25) we get

$$dP_0 = dx_0 \wedge dy_0 = \frac{2\,\omega_2 \wedge \omega_3 + \omega_2 \wedge \omega_1}{h^3}.$$

Moreover, because (5.25), we have

$$dx_1 = d\left(\frac{a+b}{c+h}\right) = \frac{2\,\omega_1 + \omega_2 + \omega_3 - \omega_4}{e\,(c+h)^2}.$$

$$dy_1 = d\left(\frac{e}{c+h}\right) = \frac{e\,(h-c)\,\omega_1 - ec\,\omega_2 + e\,(c+2h)\,\omega_3 - eh\,\omega_4}{(c+h)^2}.$$

Therefore we have

$$dP_1 = dx_1 \wedge dy_1 =$$
$$= \frac{-\,\omega_1 \wedge \omega_2 + 3\,\omega_1 \wedge \omega_3 - \omega_1 \wedge \omega_4 + 2\,\omega_2 \wedge \omega_3 - \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_4}{(c+h)^3}.$$

Thus, up to the sign, we have

$$dP_0 \wedge dP_1 = \frac{3\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4}{h^3 (c+h)^3}.$$

In virtue of (5.27) and the relation $e\,(ah - bc) = 1$ the last equality can be written

$$d\,(P_0 + P_1) = \frac{dP_0 \wedge dP_1}{y_0\, y_1\, (x_1 - x_0)^2}. \tag{5.28}$$

In order to get a geometrical interpretation of the cinematic density in terms of the coordinates of a pair of lines $G_0(p_0, \varphi_0)$, $G_1(p_1, \varphi_1)$ we proceed as follows. The lines $y = x$ and $x + y = 1$ are carried by the group (5.22) into G_0: $y = e^2 hx - e^2 b$, $G_1 : y = -e^2(c+h)x + e^2(b+a)$, and therefore we have

$$e^2 h = -\cot \varphi_0 , \quad e^2 b = -\frac{p_0}{\sin \varphi_0}, \tag{5.29}$$

$$e^2(c+h) = \cot \varphi_1 , \quad e^2(b+a) = \frac{p_1}{\sin \varphi_1}.$$

Differentiating we get

$$\frac{d\varphi_0}{\sin^2 \varphi_0} = -e^2 h\,\omega_1 + e^2 c\,\omega_2 + e^2 h\,\omega_3,$$

$$-\frac{dp_0}{\sin \varphi_0} + A\,d\varphi_0 = -e^2 b\,\omega_1 + e^2 a\,\omega_2 + e^2 b\,\omega_3,$$

$$\frac{d\varphi_1}{\sin^2 \varphi_1} = e^2(h-c)\,\omega_1 - e^2 c\,\omega_2 - e^2(2c+h)\,\omega_3 - e^2 h\,\omega_4,$$

$$\frac{dp_1}{\sin \varphi_1} + B\,d\varphi_1 = e^2(a-b)\,\omega_1 + e^2 a\,\omega_2 + e^2(2a+b)\,\omega_3 + e^2 b\,\omega_4.$$

Exterior multiplication gives

$$\frac{dG_0 \wedge dG_1}{\sin^3 \varphi_0 \sin^3 \varphi_1} = 3\,e^6\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4.$$

From (5.29) and the relation $e(ah - bc) = 1$ we deduce

$$e^3 = \frac{p_0 \cos \varphi_1 - p_1 \cos \varphi_0}{\sin \varphi_0 \sin \varphi_1}.$$

Therefore the density for pairs of lines can be written

$$d(G_0 + G_1) = \frac{dG_0 \wedge dG_1}{\sin \varphi_0 \sin \varphi_1 (p_0 \cos \varphi_1 - p_1 \cos \varphi_0)^2}.$$

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GEOMETRIE INTEGRALĂ A GRUPURILOR PROIECTIVE ALE PLANULUI CARE DEPIND DE MAI MULT DECÎT TREI PARAMETRI

Rezumat

Intr-o lucrare anterioară [5], am găsit toate grupurile proiective reale ale planului, pentru care mulţimile de puncte şi mulţimile de drepte admit o măsură invariantă. Pentru grupurile de proiectivităţi ce depind de mai mult decît trei parametri, se pot considera mulţimi de perechi de elemente (două puncte, două drepte, punct şi dreaptă) şi se pot căuta acele grupuri, care posedă o măsură invariantă pentru mulţimi de astfel de perechi de elemente. Obiectul prezentei lucrări este de a cerceta aceste cazuri. Printre altele, se obţin următoarele rezultate tipice:

a) Singurele mulţimi de perechi de elemente, care au o măsură invariantă, în raport cu grupul proiectiv general, sînt mulţimile $P+G$ (punct şi dreaptă), astfel ca P să nu fie pe G. Densitatea invariantă are forma simplă (2.14).

b) Mulţimile de perechi de puncte P_0+P_1 şi mulţimile de perechi de drepte G_0+G_1 au o densitate invariantă în raport cu grupul afin, unimodular, cu cinci parametri $(p, q, yp, xq, xp-yq)$ şi în raport cu grupul cu cinci parametri $(p, q, xq, 2xp+yq, x(xp+yq))$, dar ele nu au o măsură invariantă la grupurile proiective, cu mai mult decît cinci parametri.

c) Printre grupurile proiective, ce depind de mai mult ca trei parametri, numai grupurile (p, q, xq, xp) şi $(p, xp, yq, x(xp+yq))$ conduc la o măsură invariantă pentru mulţimi de puncte-drepte $P+G$, pentru care P aparţine lui G.

d) Singurele grupuri proiective, ce depind de mai mult ca trei parametri, care furnizează o densitate invariantă pentru mulţimi de drepte paralele sînt următoarele patru: (p, xp, q, yq), $(p, q, xq, xp+yq)$, (p, q, xp, yq), $(p, q, yp, xq, xp-yq)$.

ИНТЕГРАЛЬНАЯ ГЕОМЕТРИЯ ПРОЕКТИВНЫХ ГРУПП ПЛОСКОСТИ, ЗАВИСЯЩИХ ОТ БОЛЕЕ ЧЕМ ТРИ ПАРАМЕТРА

Краткое содержание

В работе [5], мы нашли все проективные группы плоскости, для которых множества точек и множества прямых допускают инвариантную меру. Для проективных групп, зависящих от более чем три параметра, можно учитывать множества пар элементов (две точки, две прямые, точка и прямая) и можно искать группы допускающих инвариантную меру для таких множеств пар элементов. Эти случаи рассматриваются в настоящий работе. Находится ряд результатов среди которых следующие являются типичными:

а) Единственные множества пар элементов, допускающие инвариантную меру относительно общей проективной группы, это множества $P+G$ (точка и прямая), так что P не принадлежит G. Инвариантная плотность имеет простое выражение (2.14).

б) Множества пар точек (P_0+P_1) и пар прямых (G_0+G_1) имеют инвариантную плотность относительно пятипараметрического, аффинной, унимодулярной группы $(p, q, yp, xq, xp-yq)$ относительно пятипараметрического группы $(p, q, xq, 2xp+yq, x(xp+yq))$ но не имеют инвариантной плотности относительно проективных групп зависящих от более чем пять параметров.

в) Среди проективных групп с более чем три параметра, только гриппы $(p, q, xq, xp,)$ и $(p, xp, yq, x(xp+yq))$ дают инвариантную меру для множеств $\{P+G\}$ (точка и прямая) так что P принадлежит G.

г) Единственные проективные группы с более чем три параметра. которые дают инвариантную плотность для множеств паралельных являются следующими: (p, xp, q, yq), $(p, q, xq, xp+yq)$, (p, q, xq, yq), $(p, q, yp, xq, xp-yq)$.