# INTEGRAL GEOMETRY ON SURFACES OF CONSTANT NEGATIVE CURVATURE

By L. A. Santaló

1. **Introduction.** We use the expression "integral geometry" in the sense given it by Blaschke [4]. In a previous paper [11] we generalized to the sphere many formulas of plane integral geometry and at the same time applied these to the demonstration of certain inequalities referring to spherical curves.

The present paper considers analogous questions for surfaces of constant negative curvature and consequently for hyperbolic geometry [1].

In §§2–7 we define the measure of sets of geodesic lines and the cinematic measure, making application of both in order to obtain various integral formulas such as, for example, (4.6) which generalizes a classic result of Crofton for plane geometry and (7.5) which is the generalization of Blaschke's fundamental formula of plane integral geometry.

In §8 we apply the above results to the proof of the isoperimetric property of geodesic circles (inequality (8.4)). In §9 we obtain a sufficient condition that a convex figure be contained in the interior of another, thus generalizing to surfaces of constant negative curvature a result which H. Hadwiger [8] obtained for the plane.

For what follows we must remember that on the surfaces of curvature $K = -1$ the formulas of hyperbolic trigonometry are applicable [2; 638], that is, for a geodesic triangle of sides $a$, $b$, $c$ and angles $\alpha$, $\beta$, $\gamma$, we have

$$\begin{aligned} \cosh a &= \cosh b \cosh c - \sinh b \sinh c \cos \alpha, \\ \sinh a/\sin \alpha &= \sinh b/\sin \beta = \sinh c/\sin \gamma, \\ \sinh a \cos \beta &= \cosh b \sinh c - \sinh b \cosh c \cos \alpha. \end{aligned} \tag{1.1}$$

2. **Measure of sets of geodesics.** Let us consider a surface of constant curvature $K = -1$. Let $O$ be a fixed point on that surface and $G$ a geodesic which does not pass through $O$. We know that through $O$ passes only one geodesic perpendicular to $G$ [6; 410]. Let $v$ be the distance from $O$ to $G$ measured upon this perpendicular. We shall call $\theta$ the angle which the perpendicular geodesic makes with a fixed direction at $O$. We define as the "density" to measure sets of geodesics the differential expression

$$dG = \cosh v \, dv \, d\theta, \tag{2.1}$$

that is, the measure of a set of geodesics will be the integral of the expression (2.1) extended to this set.

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To admit this definition it is necessary to prove that the measure does not depend on the point $O$ or on the direction origin of the angles $\theta$.

Let $A$ be the point in which the normal geodesic traced through $O$ cuts $G$. We shall consider another point $O_1$ and denote by $A_1$ the analogous point in which the normal traced through $O_1$ cuts $G$. Call $\alpha$ the angle formed by the geodesic $OO_1$ and the direction origin of angles at $O$; analogously, $\alpha_1$ will be the angle which the same geodesic $OO_1$ makes with the direction origin of angles at $O_1$. For brevity write

$$OA = v, \quad O_1A_1 = v_1, \quad OO_1 = \rho, \quad OA_1 = \mu, \quad O_1A = \nu, \quad AA_1 = \lambda,$$

where the left sides are the arcs of geodesics. We can also state

$$\varphi = \text{angle } OA_1O_1, \quad \psi = \text{angle } OAO_1, \quad \theta - \alpha = \text{angle } AOO_1,$$

$$\pi - (\theta_1 - \alpha_1) = \text{angle } OO_1A_1.$$

With these notations (Fig. 1) the third formula of hyperbolic geometry (1.1) applied to the triangle $OO_1A_1$ gives

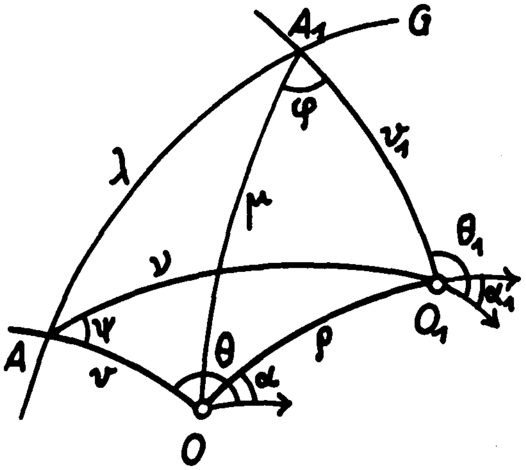


FIGURE 1

(2.2) $$\sinh \mu \cos \varphi = \cosh \rho \sinh v_1 + \sinh \rho \cosh v_1 \cos (\theta_1 - \alpha_1).$$

In the rectangular triangle $OAA_1$ the second formula (1.1) gives $\sinh \mu = \sinh v / \cos \varphi$ and in consequence (2.2) may be written as

(2.3) $$\sinh v = \cosh \rho \sinh v_1 + \sinh \rho \cosh v_1 \cos (\theta_1 - \alpha_1).$$

Analogously

(2.4) $$\sinh v_1 = \cosh \rho \sinh v - \sinh \rho \cosh v \cos (\theta - \alpha).$$

In order to pass from (2.3) to (2.4) we changed the sign of the second term of the right side because in the quadrilateral $OO_1A_1A$ the internal angle $AOO_1$ has value $\theta - \alpha$, while angle $OO_1A_1 = \pi - (\theta_1 - \alpha_1)$.

From (2.3) and (2.4) an easy calculation gives

$$\frac{\partial(v, \theta)}{\partial(v_1, \theta_1)} = \frac{\cosh^2 v_1 \sin(\theta_1 - \alpha_1)}{\cosh^2 v \sin(\theta - \alpha)}. \tag{2.5}$$

The third formula (1.1) applied to the triangle $OAA_1$ gives $\sinh \mu \sin \varphi = \cosh v \sinh \lambda$ and in the triangle $OO_1A_1$ we also have $\sinh \mu / \sin(\theta_1 - \alpha_1) = \sinh \rho / \sin \varphi$. These two equalities give

$$\cosh v \sinh \lambda = \sinh \rho \sin(\theta_1 - \alpha_1),$$

and by analogy

$$\cosh v_1 \sinh \lambda = \sinh \rho \sin(\theta - \alpha).$$

From these last equalities we deduce

$$\cosh v \sin(\theta - \alpha) = \cosh v_1 \sin(\theta_1 - \alpha_1),$$

and if we take into account this equality, the Jacobian (2.5) has the value $\cosh v_1/\cosh v$ and consequently

$$\cosh v \, dv \, d\theta = \cosh v_1 \, dv_1 \, d\theta_1 ,$$

that is, the density (2.1) and also the measure of any set of geodesics are independent of the point $O$ and the direction origin of the angles $\theta$.

3. **Measure of the geodesics which cut a line.** In the preceding section we determined the geodesic $G$ by its coördinates $v$, $\theta$. If $G$ cuts a fixed curve $C$ in a point $P$, it can also be determined by the abscissa $s$ of the point $P$ upon $C$, that is, by the length of the arc of $C$ between $P$ and an origin of arcs and the angle $\varphi$ formed at the point $P$ by the curve $C$ and the geodesic $G$. We shall express the density (2.1) as a function of $s$, $\varphi$.

Since through any point there is only one geodesic perpendicular to another geodesic, $G$ is also determined by $s$, $\theta$. We shall pass first from $v$, $\theta$ to $s$, $\theta$ and afterwards to $s$, $\varphi$. In the system of curvilinear coördinates whose curves $v$ = const. are the geodesics normal to $OA$ (the letters designate the same points as in §2) and the curves $u$ = const. are their orthogonal trajectories, we have [2; 335]

$$ds^2 = du^2 + \cosh^2 u \, dv^2. \tag{3.1}$$

Consequently, if the geodesic $G$ ($v$ = const.) forms an angle $\varphi$ with the curve $C$, we have

$$\sin \varphi = \frac{\cosh u \, dv}{ds}$$

from which

$$dv = \frac{\sin \varphi}{\cosh u} ds.$$

Therefore (2.1) may be written as

$$dG = \cosh v \frac{\sin \varphi}{\cosh u} ds\, d\theta. \tag{3.2}$$

In place of $\theta$ we wish now to introduce the angle $\varphi$. Let us call $\rho$ the arc of geodesic $OP$, $\alpha$ the angle which this arc $OP$ makes with the direction origin of angles at $O$, and $\alpha_1$ the angle which $OP$ makes with the curve $C$ (Fig. 2). In

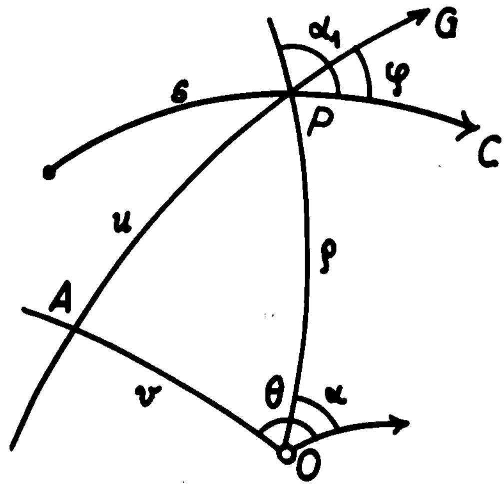


FIGURE 2

the rectangular triangle $OAP$ we have angle $AOP = \theta - \alpha$, angle $APO = \alpha_1 - \varphi$, and from (1.1) we deduce

$$\cot(\alpha_1 - \varphi) = \cosh \rho \tan(\theta - \alpha),$$

from which, by differentiation, we get

$$\frac{d\varphi}{\sin^2(\alpha_1 - \varphi)} = \frac{\cosh \rho\, d\theta}{\cos^2(\theta - \alpha)}. \tag{3.3}$$

But, in the same rectangular triangle $OAP$

$$\cos(\theta - \alpha) = \cosh u \sin(\alpha_1 - \varphi), \qquad \cosh \rho = \cosh u \cosh v$$

and as a consequence

$$\frac{\cosh \rho \sin^2(\alpha_1 - \varphi)}{\cos^2(\theta - \alpha)} = \frac{\cosh v}{\cosh u}.$$

Substituting in (3.3), we have

$$d\theta = \frac{\cosh u}{\cosh v} d\varphi, \tag{3.4}$$

and the density (3.2), after the change of the variable $\theta$ for the variable $\varphi$, will take the form

$$dG = \sin \varphi \, ds \, d\varphi. \tag{3.5}$$

This expression of the density for sets of geodesics which cut a fixed curve $C$ has the same form as for the density for straight lines of the plane [4; 13].

In order to get the measure of all the geodesics which cut a fixed curve $C$ of length $L$ we must integrate (3.5) with respect to $s$ from $Q$ to $L$ and with respect to $\varphi$ from 0 to $\pi$; the value of the integral is $2L$. In this way, if the geodesic $G$ cuts the curve $C$ in $n$ points it will be counted $n$ times. As a consequence we have

$$\int_{G \cdot C \neq 0} n \, dG = 2L, \tag{3.6}$$

the integration being extended over all the geodesics which cut the curve $C$.

We say that a closed curve $C$ is *convex* when it cannot be cut by any geodesic in more than two points. In this case, in (3.6), $n = 2$ always and we find that: *The measure of the geodesics cutting a convex curve is equal to the length of this curve.* This result and formula (3.6) have the same form for the plane [4; 11] and for the sphere [4; 81].

From (3.5) may be obtained also an integral formula which can be considered as the "dual" of (3.6). Multiplying both sides of (3.5) by the angle $\varphi$ and integrating over all values of $s$ and $\varphi$ $(0 < \varphi < \pi)$ we find that the integral of the right side has the value

$$\int_0^L ds \int_0^\pi \varphi \sin \varphi \, d\varphi = \pi L$$

and in the left side for every position of $G$ we must add the angles $\varphi_i$ $(0 < \varphi_i \leq \pi)$ which $G$ makes with $C$ at the $n$ intersections. Hence, we have

$$\int \sum_1^n \varphi_i \, dG = \pi L,$$

the integration being extended over all the geodesics which cut $C$.

4. **Density by pairs of points and integral formula for chords.** The density to measure sets of points is equal to the element of area; we shall represent it by $dP$. Let us consider a pair of points $P_1$, $P_2$. In order to measure sets of pairs of points we shall take for density the product of both densities, that is, $dP_1dP_2$. Through $P_1$ and $P_2$ only one geodesic $G$ may pass. Once this geodesic is fixed, the points $P_1$, $P_2$ are determined by their abscissa $u_1$, $u_2$ measured upon $G$ from an arbitrary origin. We wish to express the product $dP_1dP_2$ by means of $dG$ and $du_1$, $du_2$.

In a system of polar geodesic coördinates the element of arc is expressed [2; 335] by

$$ds^2 = dr^2 + \sinh^2 r\, d\varphi^2 \tag{4.1}$$

and thus the element of area is $\sinh r\, dr\, d\varphi$. When $P_1$ is fixed, the differential of the area $dP_2$ expressed in the system of polar geodesic coördinates of origin $P_1$ is

$$dP_2 = \sinh r\, dr\, d\varphi, \tag{4.2}$$

$r$ being the length of the arc of geodesic $G$ which joins $P_1$ and $P_2$, that is, $r = | u_1 - u_2 |$. From any fixed point $O$ we trace the geodesic normal to $G$ and as in §2 and §3 we call $A$ the foot of this perpendicular.

As before, we call $v$ the arc $OA$ and $\theta$ the angle made by $OA$ with a fixed direction traced by $O$. $P_1$ is supposed to be fixed. In the expression (4.2) we may introduce the angle $\theta$ in the place of the angle $\varphi$ because the relation between them is the same (3.4) already found in what precedes. Then we have

$$dP_2 = \sinh r \frac{\cosh v}{\cosh u_1} dr\, d\theta. \tag{4.3}$$

In the system of rectangular geodesic coördinates in which the curves $v =$ const. are the normal geodesics to $OA$ and the curves $u =$ const. are their orthogonal trajectories, the differential $ds$ has the form (3.1) and the differential of the area for the point $P_1$ is

$$dP_1 = \cosh u_1\, du_1\, dv_1 . \tag{4.4}$$

Taking into account (2.1) and substituting for symmetry $du_2$ instead of $dr$, from (4.3) and (4.4) we deduce

$$dP_1\, dP_2 = \sinh r\, du_1\, du_2\, dG, \tag{4.5}$$

$G$ being the geodesic which joins $P_1$ and $P_2$ and $r = | u_1 - u_2 |$ being the length of the arc $P_1P_2$. This differential formula (4.5) permits us to generalize to Crofton's formula for chords surfaces of constant negative curvature. Let $C$ be a convex curve of area $F$; $P_1$, $P_2$ two interior points to $C$; and $G$ the geodesic which joins them. We desire to integrate (4.5) over all pairs of points contained in $C$. The integral of the left side is evidently $F^2$. To calculate the integral of the right side we observe that, if $\sigma$ represents the length of the arc of the geodesic $G$ which is interior to $C$, we have

$$\int_0^\sigma \int_0^\sigma \sinh | u_1 - u_2 |\, du_1\, du_2 = 2(\sinh \sigma - \sigma)$$

and consequently

$$\int_{G \cdot C \neq 0} (\sinh \sigma - \sigma)\, dG = \tfrac{1}{2}F^2. \tag{4.6}$$

This formula (4.6) is the generalization of Crofton's formula for chords.

If $K$ is $-1/R^2$ instead of $-1$, we have

$$\int_{G\cdot C\neq 0} \left(\sinh \frac{\sigma}{R} - \frac{\sigma}{R}\right) \frac{dG}{R} = \frac{1}{2}\left(\frac{F}{R^2}\right)^2. \tag{4.7}$$

Multiplying both sides by $R^4$ and letting $R \to \infty$, we find

$$\int_{G\cdot C\neq 0} \sigma^3 \, dG = 3F^2, \tag{4.8}$$

$dG$ being the density for straight lines on the plane and $\sigma$ the length of the chord which the straight line $G$ determines in the figure plane convex $C$ of area $F$. This formula (4.8) is Crofton's integral for chords in plane geometry [4; 20], [7; 84].

5. **Density for pairs of geodesics which intersect.** If two geodesics $G_1$ and $G_2$ cut each other at a point $P$ and if $v_i$ and $\theta_i$ are the coördinates of $G_i$ ($i = 1, 2$) with respect to the origin $O$ (§2) in accordance with (2.1) we have $dG_i = \cosh v_i dv_i d\theta_i$. To measure a set of pairs of geodesics, we take the integral of the expression

$$dG_1 \, dG_2 = \cosh v_1 \cosh v_2 \, dv_1 \, d\theta_1 \, dv_2 \, d\theta_2 \, . \tag{5.1}$$

The geodesics $G_1$, $G_2$ may also be determined by their point of intersection $P$ and the angles $\varphi_1$, $\varphi_2$ which they respectively make with a fixed direction at $P$. The angle $\varphi = | \varphi_1 - \varphi_2 |$ is that formed by $G_1$ and $G_2$. If we take into account (3.1) when the geodesic $v_1$, $\theta_1$ becomes the geodesic $v_1 + dv_1$, $\theta_1$, the arc $u_1 =$ const. described by the point $P$ has the length $\cosh u_1 \, dv_1$. On the other hand, if $ds_2$ is the arc described upon $G_2$ by the intersection of $G_1$ and $G_2$, the same arc is also equivalent to $\sin \varphi \, ds_2$. Consequently,

$$\cosh u_1 \, dv_1 = \sin \varphi \, ds_2 \, , \tag{5.2}$$

and by analogy

$$\cosh u_2 \, dv_2 = \sin \varphi \, ds_1 \, . \tag{5.3}$$

Also, if we suppose $P$ fixed, the relation between the angles $\theta_i$ and $\varphi_i$ is given by (3.4), that is,

$$\cosh v_i \, d\theta_i = \cosh u_i \, d\varphi_i \, . \tag{5.4}$$

From these equalities and from (5.1) we deduce $dG_1dG_2 = \sin^2\varphi \, ds_1ds_2d\varphi_1d\varphi_2$. But $\sin \varphi \, ds_1 \, ds_2$ is equal to the element of the area $dP$ and consequently

$$dG_1 \, dG_2 = \sin \varphi \, d\varphi_1 \, d\varphi_2 \, dP. \tag{5.5}$$

This formula which expresses the product of the densities of two intersecting geodesics as a function of the density of their intersection point $P$ and the densities of the angles $\varphi_1$, $\varphi_2$ at $P$ has the same form as on the plane [4; 17] and on the sphere [4; 78].

In the cases of the plane and the sphere, as two geodesics always cut each other (the exception of straight parallel lines in the plane has no importance), integrating both sides of the formula (5.5) over all the pairs of geodesics which cut a convex curve $C$, we arrive at Crofton's fundamental formula [4; 18], [7; 78], [11]. For surfaces of constant negative curvature this reasoning cannot be applied because we may find sets of pairs of geodesics of finite measure which cut $C$ without intersecting each other in any point $P$. Nevertheless formula (5.5) is of use in obtaining the following integral formula. Let us integrate the two sides of (5.5) over all the pairs of geodesics which intersect each other in the interior of a convex curve $C$ of area $F$. The integral of the right side is

$$(5.6) \qquad \int_{P<C} dP \int_0^{\pi} \int_0^{\pi} \sin |\varphi_1 - \varphi_2| \, d\varphi_1 \, d\varphi_2 = 2\pi F.$$

To calculate the integral of the left side we first fix $G_1$. If we call $\sigma_1$ the length of the arc of $G_1$ which is inside $C$, in accordance with (3.6) the integral of $dG_2$ extended over all the $G_2$ which cut $\sigma_1$ has value $2\sigma_1$. Thus the integral of the left side of (5.5) is equivalent to $2 \int \sigma_1 dG_1$. Equating to (5.6) and writing $\sigma$ and $G$ in place of $\sigma_1$ and $G_1$, we get the integral formula

$$(5.7) \qquad \int_{G \cdot C \neq 0} \sigma \, dG = \pi F.$$

From this formula and from (4.6) we deduce

$$(5.8) \qquad \int_{G \cdot C \neq 0} \sinh \sigma \, dG = \pi F + \tfrac{1}{2} F^2.$$

In (5.8) and (5.7) as in (4.6) $\sigma$ is the length of the arc of the geodesic $G$ which is inside $C$.

6. **Cinematic measure.** Hitherto we have only considered sets of points and geodesics. Now we wish to consider sets of elements each of which is formed by a point $P$ and a direction $\varphi$ at $P$. To measure a set of such elements we take the integral of the differential form

$$(6.1) \qquad dC = dP \, d\varphi,$$

which is called cinematic density. For its definition on the plane and on the sphere, see [4; 20, 81].

On the surfaces of constant negative curvature two figures are called "congruent" if they can be superposed by a motion of the surface into itself [2; 333], [6; 409]. The position of a figure $C$ is determined by fixing an element $P$, $\varphi$ invariably bound to $C$. Consequently, the cinematic density serves also to measure any set of congruent figures.

Let $C_0$ be a fixed curve of length $L_0$ and $C$ a mobile curve of length $L$. Suppose that both curves are formed by a finite number of arcs of continuous geodesic curvature. Calling $n$ the number of intersection points of $C$ and $C_0$, a number which depends on the position of $C$, we wish to calculate

$$I = \int_{C \cdot C_0 \neq 0} n \, dC, \tag{6.2}$$

where $dC$ is the cinematic density (6.1) referred to the mobile curve $C$, and the integration is extended over all the positions of $C$.

We shall require a preliminary formula. Let $C_0$ be a fixed curve. For a point $A$ of $C_0$ consider the geodesic which makes with $C_0$ an angle $\theta$ and upon this geodesic take an arc $AA' = r$. If the point $A$ describes upon $C_0$ an arc $AB = ds_0$, $\theta$ and $r$ remaining constant, the end $A'$ will describe $A'B' = ds_0'$. Let $\theta'$ be the angle made by $AA'$ with $A'B'$. The elements $ds_0$ and $ds_0'$ may be considered to be in first approximation arcs of geodesics and accordingly we can apply formulas (1.1) of hyperbolic trigonometry. Considering only a first approximation, we have $\cosh AB = \cosh A'B' = 1$, $\sinh AB = ds_0$, $\sinh A'B' = ds_0'$. Thus the first formula (1.1) applied to the triangle $AB'A'$ gives

$$\cosh AB' = \cosh r + \sinh r \cos \theta' \, ds_0',$$

and applied to the triangle $ABB'$,

$$\cosh AB' = \cosh r + \sinh r \cos \theta \, ds_0 .$$

From these equalities it follows that

$$\cos \theta' \, ds_0' = \cos \theta \, ds_0 . \tag{6.3}$$

This is the preliminary formula sought and it is verified whether the arcs of geodesic $AA'$, $BB'$ intersect or not.

We return now to the calculation of the integral (6.2). Let $C$, $C_0$ intersect in point $A$ at angle $\alpha$ (Fig. 3). We fix at $C$ and $C_0$ an origin of arcs, $s$ being the

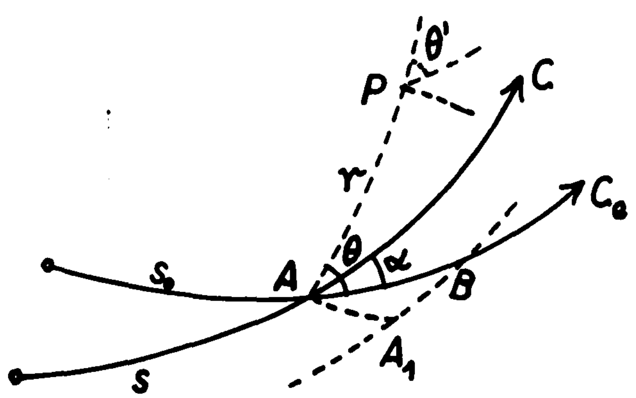


FIGURE 3

curvilinear abscissa of $A$ upon $C$ and $s_0$ the abscissa of $A$ upon $C_0$. In order to determine the position of $C$, in place of $P$, $\varphi$ which figures in (6.1), one may substitute $s$, $s_0$, $\alpha$. We wish to express the cinematic density (6.1) with these new variables $s$, $s_0$, $\alpha$. For this we must observe the following.

Let $P$ be the point invariably bound to $C$ which figures in (6.1), $PA$ the geodesic arc which unites $P$ with the intersection point $A$, and $\theta$ the angle which $PA$ makes with the fixed curve $C_0$. If $s$ and $\alpha$ are fixed and $s_0$ passes from $s_0$ to $s_0 + ds_0$, the angle $\theta$ will not vary and $P$ will describe an element of arc $ds_0'$ with value, according to (6.3),

$$ds_0' = \frac{\cos \theta}{\cos \theta'} ds_0, \tag{6.4}$$

where $\theta'$ is the angle formed by the prolongation of $AP$ with the direction of $ds_0'$. Also, $s$ and $s_0$ being fixed, when $\alpha$ passes to $\alpha + d\alpha$, the point $P$ describes an arc $ds_0''$ normal to $AP$ with value

$$ds_0'' = \sinh r \, d\alpha, \tag{6.5}$$

$r$ being the length of the arc of geodesic $PA$. This value (6.5) is obtained from the expression of the element of the arc in polar geodesic coördinates (4.1). The angle formed by the elements $ds_0'$ and $ds_0''$ is $\frac{1}{2}\pi - \theta'$ and as a consequence the element of area $dP$ expressed by the coördinates $s_0$, $\alpha$ has the value $dP = \sin(\frac{1}{2}\pi - \theta') \, ds_0' \, ds_0''$, that is, according to (6.5) and (6.4),

$$dP = \cos \theta \sinh r \, ds_0 \, d\alpha. \tag{6.6}$$

We shall suppose now that, having fixed $P$, we make $PA$ rotate, and with it all the curve $C$, through an angle $d\varphi$. The point $A$ will describe an arc $AA_1$ of a geodesic circle of center $P$, where $AA_1 = \sinh r \, d\varphi$. After turning through the angle $d\varphi$ the curve $C$ will cut $C_0$ at the point $B$ and the arc $A_1B$ is the arc $ds$ which has increased in passing from $\varphi$ to $\varphi + d\varphi$. The infinitesimal triangle $AA_1B$ may be considered a geodesic triangle and accordingly the second formula (1.1) gives

$$\frac{ds}{\cos \theta} = \frac{\sinh r \, d\varphi}{\sin(\alpha + d\alpha)},$$

that is,

$$d\varphi = \frac{\sin \alpha}{\cos \theta \sinh r} ds. \tag{6.7}$$

From (6.7), (6.6) and (6.1) we deduce

$$dC = \sin \alpha \, ds \, ds_0 \, d\alpha. \tag{6.8}$$

This is the expression sought. The angle $\alpha$ will always be considered between 0 and $\pi$.

This expression (6.8) of the cinematic density has the same form as that for the plane [4; 23] and permits the immediate calculation of (6.2). Integrating (6.8) over all the values of $s$, $s_0$, $\alpha$, we shall have the integral of $dC$ extended over all the positions in which $C$ cuts $C_0$, but if in some position $C$ and $C_0$ intersect in $n$ points, this position will have been counted $n$ times. Consequently,

$$I = \int_{C \cdot C_0 \neq 0} n \, dC = \int_0^L ds \int_0^{L_0} ds_0 \int_{-\pi}^{\pi} \sin | \alpha | \, d\alpha = 4LL_0 ,$$

that is,

$$\int_{C \cdot C_0 \neq 0} n \, dC = 4LL_0 . \tag{6.9}$$

This formula expresses the generalization to the surfaces of constant negative curvature of Poincaré's formula and it has the same form as in the case of the plane [4; 24] and the sphere [4; 81].

The expression (6.8) also permits us to obtain an integral formula which, in a certain form, is the dual formula of (6.9). If we multiply both sides of (6.8) by $\alpha$ and integrate over all the values of $s$, $s_0$, $\alpha$ $(0 < \alpha < \pi)$, the sum $\sum_1^n \alpha_i$ of the angles at which the curves $C$ and $C_0$ intersect will appear on the left side and the integral of the right side will have the value

$$\int_0^L ds \int_0^{L_0} ds_0 \int_{-\pi}^{\pi} | \alpha \sin \alpha | \, d\alpha = 2\pi LL_0 .$$

Consequently,

$$\int_{C \cdot C_0 \neq 0} \sum_1^n \alpha_i \, dC = 2\pi LL_0 . \tag{6.10}$$

It should be noted that this formula also has the same form as in the case of the plane [9; 101] and of the sphere [4; 82].

**7. Fundamental formula of cinematic measure.** On the surface of constant negative curvature $K = -1$, let us consider a closed curve $C_1$ of length $L_1$ without double points and formed by a finite number of arcs of continuous geodesic curvature. Let $F_1$ be the area bounded by $C_1$. The total geodesic curvature $K_1$ of $C_1$ is composed of the sum of the integrals $\int \kappa_g^1 \, ds_1$ of the geodesic curvature along the arcs which form $C_1$ plus the sum of the exterior angles at the angular points if these appear. Then the Gauss-Bonnet formula gives

$$K_1 = 2\pi + F_1 . \tag{7.1}$$

If $C_0$ is another closed curve of area $F_0$ with length $L_0$ and total geodesic curvature $K_0$, then $K_0 = 2\pi + F_0$ also. Suppose $C_0$ fixed and $C_1$ of variable position.

In each position of $C_1$ the intersection of the domains bounded by $C_0$ and $C_1$ will be composed of a certain number of partial domains whose boundaries are formed by arcs of $C_0$ and $C_1$. We represent by $F_{01}$ the area, by $L_{01}$ the length and by $K_{01}$ the total geodesic curvature of the domain $C_{01}$ intersection of the domains bounded by $C_0$ and $C_1$. $C_{01}$ may be multiply connected (Fig. 4).

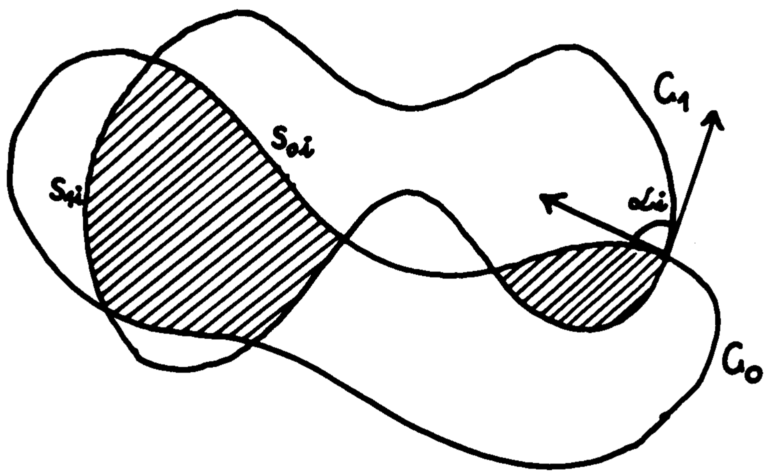


FIGURE 4

We wish to demonstrate the integral formula

$$\int_{C_0 \cdot C_1 \neq 0} K_{01}\, dC_1 = 2\pi(K_0F_1 + K_1F_0 + L_0L_1), \tag{7.2}$$

where $dC_1$ is the cinematic density (6.1) with reference to the mobile figure $C_1$, the integration being extended over all the positions of $C_1$ in which the domain bounded by this curve has any common point with that bounded by $C_0$.

Formula (7.2) is the generalization on the surfaces of curvature $K = -1$ (that is, the generalization to hyperbolic geometry) of Blaschke's fundamental formula of integral plane geometry. The proof we shall give is analogous to that given for the plane by Maak [9] and Blaschke [4; 37].

Calling $s_{0i}$, $s_{1i}$ the lengths of the arcs of $C_0$ and $C_1$ which contribute to form the boundary of $C_{01}$ and $\alpha_i$ the angles in which $C_0$ and $C_1$ intersect, by definition of $K_{01}$ we have

$$K_{01} = \sum_i \int_{s_{0i}} \kappa_g^0\, ds_0 + \sum_i \int_{s_{1i}} \kappa_g^1\, ds_1 + \sum_i \alpha_i, \tag{7.3}$$

$\kappa_g^0$ and $\kappa_g^1$ being the geodesic curvatures of $C_0$ and $C_1$. Let us consider the integral $I_0 = \int \kappa_g^0\, ds_0\, dC_1$ extended over all the positions in which the point $s_0$ belongs to the boundary of $C_0$ and is contained in the interior of $C_1$. This integral $I_0$ may be calculated in two ways. Having first fixed the point $s_0$, we must integrate $dC_1$ over all the positions of $C_1$ for which this fixed point is interior to $C_1$ ; according to (6.1) this integral has the value $2\pi F_1$ ; there

$$\int_{C_0} \kappa_g^0 \, ds_0$$

of value $K_0$ remains. Consequently, $I_0 = 2\pi F_1 K_0$ . The same integral may be calculated in another way. If we first fix $C_1$ , the integral $\int \kappa_g^0 \, ds_0$ extended over all the values of $s_0$ which are interior to $C_1$ is the sum

$$\sum_i \int_{s_{0i}} \kappa_g^0 \, ds_0$$

which appears in (7.3). We must now integrate the product of this sum by $dC_1$ . Equating this value of $I_0$ to that found before, we have

$$\int_{C_0 \cdot C_1 \neq 0} \sum_i \int_{s_{0i}} \kappa_g^0 \, ds_0 \, dC_1 = 2\pi F_1 K_0 ,$$

and analogously, for symmetry, must be

$$\int_{C_0 \cdot C_1 \neq 0} \sum_i \int_{s_{1i}} \kappa_g^1 \, ds_1 \, dC_1 = 2\pi F_0 K_1 .$$

Taking into account these values and (6.10), we have formula (7.2), which we wished to prove.

Formula (7.2) may be written in a more convenient form. For this we must calculate the integral $\int F_{01} \, dC_1$ in which $F_{01}$ is the area of the intersection of $C_0$ and $C_1$ and the integration is extended over all the positions for which $C_{01} = C_0 \cdot C_1 \neq 0$. Let us consider the integral $I_{01} = \int dP_0 \, dC_1$ , in which $dP_0$ is the element of area, extended over all the positions in which $C_1$ contains the point $P_0$ interior to $C_0$ . Having fixed $P_0$ , we find that the integral of $dC_1$ has a value $2\pi F_1$ and when $P_0$ is varied over all the interior of $C_0$ we obtain $I_{01} = 2\pi F_0 F_1$ . Also if we fix $C_1$ first, the point $P_0$ can vary over all the points of the intersection of $C_0$ and $C_1$ . The integral of $dP_0$ will then be $F_{01}$ and consequently $I_{01} = \int F_{01} \, dC_1$ . Equating the two values obtained for $I_{01}$ , we have

$$\int_{C_0 \cdot C_1 \neq 0} F_{01} \, dC_1 = 2\pi F_0 F_1 . \tag{7.4}$$

By the Gauss-Bonnet theorem, if the intersection of $C_0$ and $C_1$ is composed of $\nu$ simply connected pieces (for example, in Fig. 4, $\nu = 2$), we have $K_{01} = 2\pi\nu + F_{01}$ and moreover $K_0 = 2\pi + F_0$ , $K_1 = 2\pi + F_1$ . Substituting these values in (7.2) and taking into account (7.4), we find

$$\int_{C_0 \cdot C_1 \neq 0} \nu \, dC_1 = 2\pi(F_0 + F_1) + F_0 F_1 + L_0 L_1 . \tag{7.5}$$

In particular, if $C_0$ and $C_1$ are convex, their intersection is always simply connected, that is, composed of only one piece. As a consequence $\nu = 1$ and we have the result:

*The measure of the positions of a convex figure $C_1$ in which it has some common point with another convex figure $C_0$ has the value*

$$\int_{C_0 \cdot C_1 \neq 0} dC_1 = 2\pi(F_0 + F_1) + F_0F_1 + L_0L_1 . \tag{7.6}$$

In the following sections we shall apply this formula and (6.9).

8. **Isoperimetric propriety of geodesic circles.** On the surface of constant negative curvature $K = -1$, let us consider a closed curve $C$ which has no double points, and which has length $L$ and area $F$. We consider the set of curves congruent to $C$ which have points in common with $C$. Calling $M_i$ the cinematic measure of the set of these curves which have $i$ points in common with $C$, we can write formula (7.6) as

$$M_2 + M_4 + M_6 + \cdots = 4\pi F + F^2 + L^2, \tag{8.1}$$

since now $C_0 = C_1 = C$. Analogously, formula (6.9) gives

$$2M_2 + 4M_4 + 6M_6 + \cdots = 4L^2. \tag{8.2}$$

From these two equalities we deduce

$$L^2 - F^2 - 4\pi F = M_4 + 2M_6 + 3M_8 + \cdots \tag{8.3}$$

and, as the $M_i$, which are the measure of certain sets, are always non-negative, we have

$$L^2 - F^2 - 4\pi F \geq 0. \tag{8.4}$$

This is the isoperimetric inequality on surfaces of constant negative curvature $K = -1$. In fact, from (8.4) can be deduced that for all the curves which limit an area $F$ the minimum value of the length is $(F^2 + 4\pi F)^{\frac{1}{2}}$. This minimum value $L_0$ is reached by the geodesic circles. Hence if $C_0$ is a geodesic circle of radius $\rho_0$ we have [6; 404]

$$L_0 = 2\pi \sinh \rho_0 , \qquad F_0 = 2\pi(\cosh \rho_0 - 1) \tag{8.5}$$

and therefore $L_0^2 = F_0^2 + 4\pi F_0$.

This proof of the isoperimetric inequality (8.4) does not permit the assertion that the geodesic circles are the only figures for which the equality in (8.4) is valid. For this we shall give another proof leading to an inequality stronger than (8.4).

Let $\rho_0$ have such a value that no geodesic circle of radius $\rho_0$ is contained in the interior of $C$ nor contains $C$ in its own interior. Also let $C_0$ be the geodesic circle of radius $\rho_0$. Calling $M_i$ the measure of the set of circles $C_0$ which intersect $C$ in $i$ points, in accordance with (7.6) and (6.9) we have

$$M_2 + M_4 + M_6 + \cdots = 2\pi(F + F_0) + FF_0 + LL_0, \tag{8.6}$$

and

$$2M_2 + 4M_4 + 6M_6 + \cdots = 4LL_0. \tag{8.7}$$

From these equalities we deduce

$$LL_0 - FF_0 - 2\pi(F + F_0) = M_4 + 2M_6 + 3M_8 + \cdots \geq 0. \tag{8.8}$$

To abbreviate, we put

$$\Delta = L^2 - F^2 - 4\pi F, \tag{8.9}$$

where $\Delta$ is the "isoperimetric deficit". $F_0$ and $L_0$ given by (8.5) thus satisfy

$$L_0^2 - F_0^2 - 4\pi F_0 = 0, \tag{8.10}$$

with which we easily prove the identity

$$\frac{1}{2FF_0}[\Delta F_0^2 - (LF_0 - FL_0)^2] = LL_0 - FF_0 - 2\pi(F + F_0). \tag{8.11}$$

Taking into account (8.8), we deduce from (8.11) that

$$\Delta \geq \frac{1}{F_0^2}(LF_0 - FL_0)^2,$$

or by substituting for $F_0$, $L_0$ their values (8.5), we get

$$\Delta \geq (L - F \coth \tfrac{1}{2}\rho_0)^2. \tag{8.12}$$

This inequality is verified for any $\rho_0$ so that no circle of radius $\rho_0$ could contain $C$ or itself be contained in $C$. In particular, if $\rho_e$ is the minimum radius of the geodesic circles which enclose $C$ and $\rho_i$ the maximum of those contained in the interior of $C$, we have

$$\Delta \geq (L - F \coth \tfrac{1}{2}\rho_e)^2, \qquad \Delta \geq (F \coth \tfrac{1}{2}\rho_i - L)^2, \tag{8.13}$$

and taking into account the inequality

$$x^2 + y^2 \geq \tfrac{1}{2}(x + y)^2, \tag{8.14}$$

from (8.13) we deduce

$$\Delta \geq \tfrac{1}{4}F^2(\coth \tfrac{1}{2}\rho_i - \coth \tfrac{1}{2}\rho_e)^2. \tag{8.15}$$

Analogously, taking into account (8.10), we easily prove the identity

$$\frac{4\pi + F_0}{2L_0^2(4\pi + F)}[\Delta L_0^2 - ((4\pi + F)F_0 - LL_0)^2] = LL_0 - FF_0 - 2\pi(F + F_0), \tag{8.16}$$

and consequently, in accordance with (8.8), we have

$$\Delta \geq \frac{1}{L_0^2}((4\pi + F)F_0 - LL_0)^2. \tag{8.17}$$

If we substitute the values (8.5), we may write this inequality

$$\Delta \geq ((4\pi + F) \tanh \tfrac{1}{2}\rho_0 - L)^2. \tag{8.18}$$

Writing this inequality for $\rho_e$ and $\rho_i$ and taking into account (8.14), we deduce

$$\Delta \geq \tfrac{1}{4}(4\pi + F)^2(\tanh \tfrac{1}{2}\rho_e - \tanh \tfrac{1}{2}\rho_i)^2. \tag{8.19}$$

The isoperimetric inequalities (8.15) and (8.19) are stronger than (8.4). They make clear that the equality $\Delta = 0$ can be verified only when $\rho_i = \rho_e$, that is, when $C$ is a geodesic circle. It is thus completely proved that on surfaces of constant negative curvature the geodesic circles are the only curves which for a specified length enclose maximum area. A direct proof of the inequalities (8.15) and (8.19) was given by us in [12]. The isoperimetric problem on the surfaces of constant negative curvature has also been solved in a completely distinct manner in [13].

For a surface of constant curvature $K = -1/R^2$, the inequalities (8.15) and (8.19) are written respectively

$$\left(\frac{L}{R}\right)^2 - \left(\frac{F}{R^2}\right)^2 - 4\pi\frac{F}{R^2} \geq \frac{1}{4}\frac{F^2}{R^2}\left(\frac{1}{R}\coth\frac{\rho_i}{2R} - \frac{1}{R}\coth\frac{\rho_e}{2R}\right)^2,$$

$$\left(\frac{L}{R}\right)^2 - \left(\frac{F}{R^2}\right)^2 - 4\pi\frac{F}{R^2} \geq \frac{1}{4}\left(4\pi + \frac{F}{R^2}\right)^2\left(\tanh\frac{\rho_e}{2R} - \tanh\frac{\rho_i}{2R}\right)^2.$$

Multiplying by $R^2$ and making $R \to \infty$, we obtain

$$L^2 - 4\pi F \geq F^2\left(\frac{1}{\rho_i} - \frac{1}{\rho_e}\right)^2 \tag{8.20}$$

and

$$L^2 - 4\pi F \geq \pi^2(\rho_e - \rho_i)^2, \tag{8.21}$$

which are isoperimetric inequalities for plane figures. In these, $\rho_i$ is the maximum radius of those circles which are contained in $C$ and $\rho_e$ the minimum of these which contain $C$. Inequality (8.21) is a classic inequality due to Bonnesen [5; 63].

**9. A sufficient condition that a convex curve congruent to $C$ be contained in the interior of another convex curve $C_0$.** Let $C_1$ be a convex curve of length $L_1$ which limits a domain of area $F_1$. $C_0$ is another convex curve of length $L_0$ and area $F_0$. We wish to find a sufficient condition that a curve congruent to $C_1$ be contained in the interior of $C_0$. As in §8, let $M_i$ be the measure of the set of curves congruent to $C_1$ which intersect $C_0$ in $i$ points. $M_0$ is the measure of the set of curves congruent to $C_1$ which are in the interior of $C_0$ or which contain $C_0$.

According to (7.6) and (6.9) we have

$$(9.1)\qquad M_0 + M_2 + M_4 + M_6 + \cdots = 2\pi(F_0 + F_1) + F_0F_1 + L_0L_1$$

and

$$(9.2)\qquad 2M_2 + 4M_4 + 6M_6 + \cdots = 4L_0L_1 .$$

From these equalities

$$(9.3)\qquad 2\pi(F_0 + F_1) + F_0F_1 - L_0L_1 = M_0 - M_4 - 2M_6 - \cdots .$$

If $M_0 = 0$, the left side of this equality is non-positive. Hence a sufficient condition that a curve congruent to $C_1$ contain $C_0$ or be contained in $C_0$ is

$$(9.4)\qquad 2\pi(F_0 + F_1) + F_0F_1 - L_0L_1 > 0.$$

In order to sharpen this result, we first observe that by (8.4) for any $C_0$ and $C_1$

$$L_0^2 \geq F_0^2 + 4\pi F_0 , \qquad L_1^2 \geq F_1^2 + 4\pi F_1$$

and hence

$$(9.5)\qquad L_0^2L_1^2 \geq F_0F_1(4\pi + F_0)(4\pi + F_1).$$

Consider the inequality

$$(9.6)\qquad L_0L_1 - F_1(4\pi + F_0) > [L_0^2L_1^2 - F_0F_1(4\pi + F_0)(4\pi + F_1)]^{\frac{1}{2}},$$

whose right side is always real by (9.5). If we square and simplify (9.6), we see that (9.4) is also verified. Consequently, one of the two curves $C_0$ or $C_1$ can be contained in the interior of the other. We shall prove that if (9.6) is verified, $F_1 < F_0$ and consequently $C_1$ can be contained in $C_0$ . In fact, if $F_1 \geq F_0$ , in accordance with (9.4), which is a consequence of (9.6), we have $L_0L_1 < 4\pi F_1 + F_0F_1$ and the inequality (9.6) is not verified since the left side must be positive. Consequently,

*$C_0$ and $C_1$ being two convex curves on the surface of curvature $K = -1$, the inequality* (9.6) *is a sufficient* (*but not necessary*) *condition that a curve congruent to $C_1$ be contained in the interior of $C_0$ .*

In particular, if $C_1$ is a geodesic circle of radius $\rho_1$ , taking into account the values (8.5), we can write the inequality (9.6) as

$$(9.7)\qquad L_0 - (4\pi + F_0) \tanh \tfrac{1}{2}\rho_1 > (L_0^2 - F_0(4\pi + F_0))^{\frac{1}{2}}$$

and this inequality is a sufficient condition that $C_0$ contain in its interior a geodesic circle of radius $\rho_1$ .

Analogously,

$$(9.8)\qquad L_1 - F_1 \coth \tfrac{1}{2}\rho_0 > (L_1^2 - F_1(4\pi + F_1))^{\frac{1}{2}}$$

is a sufficient condition that $C_1$ be contained in the interior of a geodesic circle of radius $\rho_0$ .

If the curvature of the surface is $K = -1/R^2$, the corresponding condition (9.6) can be written without difficulty. Multiplying both sides by $R^2$ and making $R \rightarrow \infty$, we obtain

$$L_0L_1 - 4\pi F_1 > (L_0^2L_1^2 - 16\pi^2F_0F_1)^{\frac{1}{2}}, \tag{9.9}$$

which is a sufficient condition that a plane convex curve congruent to $C_1$ of area $F_1$ and length $L_1$ be contained in the interior of $C_0$ whose area and length are $F_0$ and $L_0$ respectively.

Condition (9.9) for the plane has been obtained by H. Hadwiger [8]. For the analogous condition for the curves on the sphere see [10].

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Mathematical Institute, Rosario, Argentina.