MEASURE OF SETS OF GEODESICS IN A RIEMANNIAN SPACE AND APPLICATIONS TO INTEGRAL FORMULAS IN ELLIPTIC AND HYPERBOLIC SPACES (*)

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1. Introduction

The Integral Geometry on surfaces has been considered by Blaschke [2] ,Haimovici [5], Vidal Abascal [9] and the author [7]. In § 2, 3, 4, 5 of the present paper we consider some points of the Integral Geometry in a Riemannian n-dimensional space. We start with the definition of density for sets of geodesics and obtain some integral formulas (for instance (3.2) and (5.3)) which generalize to Riemannian spaces well known results of the Euclidean space.

In § 6 we consider, as particular cases, the elliptic and hyperbolic spaces and we give some integral formulas referring to convex bodies in these spaces.

In § 7 the elliptic space is considered in more detail. The "duality" which holds in this space permits the obtention of some more integral formulas.

In what follows ω_i shall represent the area of the Euclidean i-dimensional unit sphere and χ_i the volume bounded by it, that is,

$$(1.1) \qquad \omega_i = \frac{2\,\pi^{(i+1)/2}}{\Gamma(i+1)/2} \quad , \quad \chi_i = \frac{2\,\pi^{(i+1)/2}}{(i+1)\Gamma(i+1)/2)}$$

2. Density for sets of geodesics in a n-dimensional Riemannian space

Let R_n be a n-dimensional Riemannian space defined by

$$(2.1) \qquad ds^2 = g_{ij}\, dx^i dx^j$$

and let us introduce the following notations

$$(2.2) \qquad F = (g_{ij}\dot{x}^i\dot{x}^j)^{1/2} \quad , \quad p_i = \partial F/\partial \dot{x}^i.$$

The density for sets of geodesics is the following exterior differential form, *taken always in absolute value,*

$$(2.3) \quad dG = \sum_{i=1}^{n} [dp_1 dx^1 \ldots dp_{i-1} dx^{i-1} dp_{i+1} dx^{i+1} \ldots dp_n dx^n].$$

The measure of a set of geodesics will be the integral of dG extended over the set.

The density (2.3) is the $(n-1)$th power of the exterior diffetential invariant form $\sum [dp_i\, dx^i]$ which integral constitutes the invariant integral of Poincaré of the dynamics [3, p. 19,78]. Therefore it possesses the following two fundamental properties of invariance: a) It is invariant with respect to a change of coordinates in the space; b) It is invariant under displacements of the elements $(x^i,\ p_i)$ on the respective geodesic.

In order to give a geometrical interpretation of the density dG let us consider a fixed hypersurface S^{n-1} and a set of geodesics which intersect S^{n-1}. Let G be such a geodesic and P its intersection point with S^{n-1}. In a neighborhood of P we may assume that the equation of S^{n-1} is $x^n = 0$ and that the coordinate system is orthogonal, that is

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \ldots + g_{nn}(dx^n)^2$$

and thus

$$p_i = g_{ii} \frac{dx_i}{ds}.$$

If α^i represents the cosine of the angle between G and the x^i-coordinate curve at P, we have

$$\alpha^i = \sqrt{g_{ii}}\, \frac{dx^i}{ds}$$

and

$$(2.4) \qquad p_i = \sqrt{g_{ii}}\alpha^i \quad , \quad dp_i = \sqrt{g_{ii}}d\alpha^i + \frac{\partial \sqrt{g_{ii}}}{\partial x^h}\alpha^i \, dx^h .$$

In order to determine G, acording to the property b) of invariance of dG, we may choose its intersection point P with S^{n-1}. At the point P it is $x^n = 0$, $dx^n = 0$ and consequently the density (2.3) takes the form

$$dG = [dp_1 \, dx^1 \ldots dp_{n-1} \, dx^{n-1}]$$

or, according to (2.4),

$$(2.5) \qquad dG = (g_{11}g_{22}\ldots g_{n-1,\,n-1})^{1/2} [d\alpha^1 \, d\alpha^2 \ldots d\alpha^{n-1}dx^1 \ldots dx^{n-1}].,$$

If $d\sigma$ represents the element of $(n-1)$-dimensional area on S^{n-1} we have

$$d\sigma = (g_{11} \, g_{22} \ldots g_{n-1,\,n-1})^{1/2} \, dx^1 \, dx^2 \ldots dx^{n-1}.$$

On the other hand the element of area on the $(n-1)$-dimensional unit sphere of center P corresponding to the direction of the tangent to G at P has the value

$$(2.6) \qquad d\omega_{n-1} = \frac{[d\alpha^1 \ldots d\alpha^{n-1}]}{|\alpha^n|} .$$

Hence we have

$$(2.7) \qquad dG = |\alpha^n| \, [d\omega_{n-1}d\sigma] = |\cos \varphi| \, [d\omega_{n-1}d\sigma]$$

where $\alpha^n = \cos \varphi$ is the cosine of the angle φ between the tangent to G and the normal to S^{n-1} at the point P.

3. Geodesics which intersect a fixed hypersurface

The expression (2.7) of dG gives immediately a very general integral formula. Let $f(\sigma, \varphi)$ be an integrable function defined on S^{n-1} depending upon the point $P(\sigma)$ und upon the direction φ at it. Multiplying both sides of (2.7) by $f(\sigma, \varphi)$ and performing the integration over the hypersurface S^{n-1} and the half of the $(n-1)$-dimensional unit sphere (in order to consider non-oriented geodesics),

in the left side each geodesic G appears as common factor of the sum $\&\, f(\sigma_i, \varphi_i)$ of the values of $f(\sigma, \varphi)$ at the m intersection points of G with S^{n-1}. Consequently we have

$$(3.1) \qquad \int \sum_1^m f(\sigma_i, \varphi_i)\, dG = \int_{S^{n-1}} \int_{\frac{1}{2}\omega_{n-1}} f(\sigma, \varphi)\, |\cos \varphi|\, d\sigma\, d\omega_{n-1}.$$

For instance, if $f(\sigma, \varphi) = 1$, the integral of $|\cos \varphi|\, d\omega_{n-1}$ gives a half of the projection of the $(n-1)$-dimensional unit sphere upon a diametral plane; consequently we get

$$(3.2) \qquad\qquad \int m\, dG = \chi_{n-2}\, F$$

where χ_{n-2} is given by (1.1) and m denotes the number of intersection points of G and S^{n-1}. The integral is extended over all geodesics which intersect S^{n-1} and F represents the area of S^{n-1}.

4. Convex domains

We shall say that a simple closed hypersurface S^{n-1} is *convex* when any geodesic which intersects it has either two points or a whole arc in common with S^{n-1}. In this case, if S^{n-1} has a finite area F, the measure of the geodesics which have a common arc with S^{n-1} is zero and consequently (3.2) gives: *the measure of the geodesics cutting a convex hypersurface of area F is equal to $\frac{1}{2} \chi_{n-2}F$.*

A domain Q in our Riemannian space will be said to be convex when the following three properties are satisfied: 1. It is bounded by a closed convex hypersurface; 2. It is homeomorphic to a $(n-1)$-dimensional sphere; 3. Every geodesic with a point P interior to Q can be prolonged from P in both senses to points outside Q.

Let S^{n-1} be a hypersurface of area F contained in the interior of a convex domain Q; let F_o be the area of the boundary S_o^{n-1} of Q. According to the foregoing condition 3 every geodesic which intersects S^{n-1} will also intersect S_o^{n-1}. Consequently (3.2) gives the

following mean value for the number of intersection points of S^{n-1} and all geodesics which cut S_c^{n-1}

$$(4.1) \qquad m^* = \int m \, dG \Big/ \int dG = 2 F/F_o.$$

As an immediate consequence we have the theorem

Given a hypersurface of area F contained inside a convex domain bounded by a hypersurface of area F_o, there exist geodesic lines which intersect S^{n-1} in a number of points $\geq 2 F/F_o$.

Exactly the same method applied to (3.1), yields the more general theorem:

Given a hypersurface S^{n-1} contained inside a convex domain bounded by a hypersurface of area F_o and an integrable function $f(\sigma, \varphi)$ depending upon the points $P(\sigma)$ of S^{n-1} und upon the angles φ around the normal to S^{n-1} at P, there exist geodesic lines G for whose intersection points $P_i = P(\sigma_i)$ $(i = 1, 2, \ldots, m)$ with S^{n-1} the relation

$$(4.2) \qquad \sum_1^n f(\sigma_i, \varphi_i) \geq \frac{2}{\chi_{n-2} F_o} \int_{S^{n-1}} \int_{\frac{1}{2}\omega^{n-1}} f(\sigma, \varphi) |\cos\varphi| \, d\sigma \, d\omega_{n-1}$$

holds, where φ_i is the angle at P_i between G and the normal to S^{n-1}.

5. Sets of geodesic segments

Let t be the arc length on the geodesic G. From (2.7) we deduce

$$(5.1) \qquad [dG \, dt] = |\cos \varphi| [d\omega_{n-1} d\sigma \, dt].$$

The product $|\cos \varphi| \, dt$ equals the projection of the arc element dt upon the normal to the hypersurface S^{n-1} at the point P. Consequently $|\cos \varphi| \, d\sigma \, dt$ represents the element of volume dP of the given Riemannian space at P. Consequently (5.1) may be written in the form

$$(5.2) \qquad [dG \, dt] = [dP \, d\omega_{n-1}].$$

An oriented segment S of geodesic can be determined either by G, t (G = geodesic which contains S, t = abscissa on G of the origin

of S) or by P, ω_{n-1} (P = origin of S, ω_{n-1} = point on the unit sphere which gives the direction of S). The two equivalent differential forms (5.2) may therefore be taken as density for sets of segments of geodesic lines.

Let us consider the measure of the set of oriented segments S with the origin inside a fixed domain D. The integral of the left hand side of (5.2) gives $2 \int \sigma dG$ where σ denotes the lenght of the arc of G which lies inside D (the factor 2 appears as a consequence that dG means the density for non-oriented geodesic lines). The integral of the right side of (5.2) is equal to $\omega_{n-1}V$, where V is the volume of D. Consequently we have the following integral formula

$$(5.3) \qquad \int \sigma dG = \tfrac{1}{2} \omega_{n-1} V$$

which for $n = 2, 3$ generalizes well known results of the integral geometry of the Euclidean spaces.

6. An integral formula for convex bodies in spaces of constant curvature

Let R_n be now a Riemannian space of constant curvature K. With respect to a system of polar coordinates it is known that the element of length can be written in the form [4, p. 240],

$$(6.1) \qquad ds^2 = d\rho^2 + \frac{\operatorname{sen}^2 \sqrt{K}\rho}{K} d\lambda^2_{n-1}$$

where ρ denotes the geodesic distance from a fixed point (origin of coordinates) and $d\lambda_{n-1}$ represents the element of length of the $(n-1)$-dimensional unit sphere. The element of volume will take the form

$$(6.2) \qquad dP = \frac{\operatorname{sen}^{n-1} \sqrt{K}\rho}{K^{(n-1)/2}} [d\rho \, d\omega_{n-1}]$$

where $d\omega_{n-1}$ denotes the element of area on the $(n-1)$-dimensional unit sphere.

Let P_1, P_2 be two points in R_n and let G be the geodesic which unites them. Let ρ_1, ρ_2 be the abscissas on G of P_1 and P_2. With respect to a system of geodesic polar coordinates with the origin at P_1, the element of volume dP_2 has the form

$$(6.3) \qquad dP_2 = \frac{\operatorname{sen}^{n-1}\sqrt{K}\,|\rho_2-\rho_1|}{K^{(n-1)/2}}\,[d\rho_2\,d\omega_{n-1}]\,.$$

By exterior multiplication by dP_1 we have in consequence of (5.2)

$$(6.4) \qquad [dP_1\,dP_2] = \frac{\sin^{n-1}\sqrt{K}\,|\rho_2-\rho_1|}{K^{(n-1)/2}}\,[d\rho_1\,d\rho_2\,dG].$$

This formula was given following different way by HAIMOVICI [6].

Let us consider the case $n = 3$. If Q is a convex domain of volume V and we consider all the pairs of points P_1, P_2 inside Q, the integral of the left side of (6.4) is equal to V^2. If σ denotes the length of the arc of G which lies inside Q, by calculating the integral of the right side, we have

$$\int_0^\sigma \int_0^\sigma \sin^2\sqrt{K}\,|\rho_2-\rho_1|\,d\rho_2 d\rho_1 = \tfrac{1}{2}\left(\sigma^2 - \frac{1}{K}\sin^2\sqrt{K}\sigma\right)$$

Hence we have the integral formula

$$(6.5) \qquad \frac{1}{K}\int\left(\sigma^2 - \frac{1}{K}\sin^2\sqrt{K}\,\sigma\right)dG = 2\,V^2$$

where the integral is extended over all the geodesics which intersect Q.

For the elliptic space ($K = 1$) this formula reduces to

$$(6.6) \qquad \int(\sigma^2 - \sin^2\sigma)\,dG = 2\,V^2\,,$$

and for the hyperbolic space ($K = -1$)

(6.7)
$$\int (\text{sh}^2\sigma - \sigma^2) \, dG = 2\, V^2 .$$

For the Euclidean space $(K = 0)$ we observe that

$$\lim_{K \to 0} \frac{1}{K} \int \left(\sigma^2 - \frac{1}{K} \left(\sqrt{K}\sigma - \frac{K\sqrt{K}}{3!}\sigma^3 + \dots \right)^2 \right) dG = \frac{1}{3} \int \sigma^4 dG$$

and consequently we have

(6.8)
$$\int \sigma^4 dG = 6\, V^2 ,$$

which is a well known formula [1, p. 77].

7. Integral formulas for convex bodies in the elliptic space

In the elliptic n-dimensional space all geodesics are closed and have the finite length π. The hyperplanes have finite area $\frac{1}{2}\omega_{n-1}$. Since any geodesic intersects a fixed hyperplane in one and only one point, formula (3.2) gives the measure of the set of all geodesics of the n-dimensional elliptic space:

(7.1)
$$\int dG = \tfrac{1}{2} \chi_{n-2}\, \omega_{n-1}.$$

Let Q be a convex body of area F and volume V and let us consider the set of geodesic segments of length π which intersect Q. The integral of the left side of (5.2) extended over this set is

(7.2)
$$\int dG \, dt = \pi \int dG = \tfrac{1}{2} \pi \chi_{n-2}\, F$$

and the integral of the right side of (5.2) is

(7.3)
$$\int dP d\omega_{n-1} = \tfrac{1}{2} \omega_{n-1}\, V + \int \Omega dP$$

where Ω denotes the angle under which Q is seen from P (P exterior to Q). From (7.2) and (7.3) we deduce

(7.4)
$$\int \Omega \, dP = \tfrac{1}{2} \pi \chi_{n-2}\, F - \tfrac{1}{2} \omega_{n-1}\, V.$$

For instance for $n = 2$ we get the known formula [8],

$$(7.5) \qquad \int \Omega \, dP = \pi L - \pi F$$

where L denotes the length of Q and F its area.

For $n = 3$ we have

$$(7.6) \qquad \int \Omega \, dP = \tfrac{1}{2} \pi^2 F - 2 \pi V .$$

In the elliptic space to each integral formula referring to convex bodies corresponds another one by "duality". For the sake of simplicity we shall consider the case $n = 3$; the case $n = 2$ was already considered in [8].

Let M, F, V be the integrated mean curvature, area and volume of a given convex body Q. For the dual conex body Q^* it is known that we have

$$(7.7) \qquad F^* = 4\pi - F, \quad M^* = M \quad , \quad V^* = \pi^2 - M - V .$$

By duality to each straight line G corresponds another straight line G^* and hence, having into account (7.7), the formula (5.3) writes

$$\int (\pi - \varphi^*) \, dG^* = 2 \pi (\pi^2 - M^* - V^*)$$

where φ^* denotes the angle between the two support planes to Q through G^* and the integral is extended over all G^* exterior to Q. Having into account (7.1) and (3.2), and by replacing G^* by G, we have the integral formula

$$(7.8) \qquad \int \varphi \, dG = 2 \pi (M + V) - \tfrac{1}{2} \pi^2 F$$

which has no analogous in the Euclidean geometry.

Let us now consider the formula (6.6). Applied to the dua convex body Q^* we have

$$\int [(\pi - \varphi^*)^2 - \sin^2 \varphi^*] \, dG^* = 2 (\pi^2 - M^* - V^*)^2$$

from which and (7.8), (7.1), (3.2) it follows that

$$(7.9) \qquad \int (\varphi^2 - \sin^2 \varphi)\, dG = 2\, (M + V)^2 - \tfrac{1}{2}\, \pi^3 F$$

where, as in (7.8), φ denotes the angle between the two planes of support through G and the integral is extended over all G exterior to Q.

For the elliptic space of curvature $K = 1/R^2$ the formula (7.9) becomes

$$\int (\varphi^2 - \sin^2 \varphi)\, \frac{dG}{R^2} = 2 \left(\frac{M}{R} + \frac{V}{R^3} \right)^2 - \tfrac{1}{2}\, \pi^3\, \frac{F}{R^2}$$

and by $R \to \infty$, after multiplication by R^2, we get

$$\int (\varphi^2 - \text{sen}^2\, \varphi)\, dG = 2\, M^2 - \tfrac{1}{2}\, \pi^3\, F$$

which is the well known formula due to HERGLOTZ which corresponds to (7.9) for the 3-dimensional Euclidean space [1, p. 79].

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