Reprinted from Vol. I, PROCEEDINGS OF THE
INTERNATIONAL CONGRESS OF MATHEMATICIANS, 1950
Printed in U.S.A.

INTEGRAL GEOMETRY IN GENERAL SPACES

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Let E be a space of points in which a locally compact group of transformations G operates transitively. Let dx be the left invariant element of volume in G. Let H_0 and K_0 be two sets of points in E and denote by xH_0 the transformed set of H_0 by x ($x \in G$). Let us first assume that the identity is the only one transformation of G which leaves H_0 invariant. If $F(K_0 \cap xH_0)$ is a function of the intersection $K_0 \cap xH_0$, the main purpose of the so-called Integral Geometry (in the sense of Blaschke) is the evaluation of integrals of the type

$$I = \int_G F(K_0 \cap xH_0)\, dx \tag{1}$$

and to deduce from the result some geometrical consequences for the sets K_0 and H_0.

Let us now suppose that there is a proper closed subgroup g of G which leaves H_0 invariant. The elements $H = xH_0$ will then be in one to one correspondence with the points of the homogeneous space G/g. If there exists in G/g an invariant measure and dH denotes the corresponding element of volume, the Integral Geometry also deals with integrals of the type

$$I = \int_{G/g} F(K_0 \cap H)\, dH \tag{2}$$

from which it tries to deduce geometrical consequences for K_0.

In what follows we shall give some examples and applications of the method.

1. Immediate examples. Let us assume G compact and therefore of finite measure which we may suppose equal 1. In order to define a measure $m(K_0)$ of a set of points K_0, invariant with respect to G, we choose a fixed point P_0 in E and set

$$m(K_0) = \int_G \varphi(x)\, dx \tag{3}$$

where $\varphi(x) = 1$ if $xP_0 \in K_0$ and $\varphi(x) = 0$ otherwise.

If the measures $m(K_0)$, $m(H_0)$, and $m(K_0 \cap xH_0)$ exist, it is then known and easy to prove that

$$\int_G m(K_0 \cap xH_0)\, dx = m(K_0)m(H_0), \tag{4}$$

and since $\int_G dx = 1$, the mean value of $m(K_0 \cap xH_0)$ will be $m(K_0)m(H_0)$. Therefore we have: *Given in E two sets K_0, H_0, there exists a transformation x of G such that $m(K_0 \cap xH_0)$ is equal to or greater than $m(K_0)m(H_0)$.*

If K_0 consists of N points P_i ($i = 1, 2, \cdots, N$) and call $\nu(K_0 \cap xH_0)$ the number of points P_i which belong to xH_0, we want to evaluate $\int_G \nu(K_0 \cap xH_0)\, dx$. We set $\varphi_i(x) = 1$ if $xP_i \in H_0$ and $\varphi_i(x) = 0$ otherwise. According to (3) and the invariance of dx we have

$$m(H_0) = \int_G \varphi_i(x)\, dx = \int_G \varphi_i(x^{-1})\, dx$$

where $\varphi_i(x^{-1}) = 1$ if $P_i \in xH_0$ and $\varphi_i(x^{-1}) = 0$ otherwise. Consequently we have

$$\int_G \nu(K_0 \cap xH_0)\, dx = \sum_1^N \int_G \varphi_i(x^{-1})\, dx = Nm(H_0). \tag{5}$$

Thus the mean value of ν is equal to $Nm(H_0)$ and we have: *Given N points P_i in E and a set H_0 of measure $m(H_0)$, there exists a transformation x of G such that xH_0 contains at least $Nm(H_0)$ of the given points; it contains certainly a number greater than $Nm(H_0)$ if H_0 is closed.*

2. An application to convex bodies. Let E be now the euclidean 3-space and G the group of the unimodular affine transformations which leave invariant a fixed point O. Let H be the planes of E. The subgroup g will consist of all affinities of G which leave invariant a fixed plane H_0. Each plane H can be determined by its distance p to O and the element of area $d\omega_2$ on the unit 2-sphere corresponding to the point which gives the direction normal to H. The invariant element of volume in G/g is then given by

$$dH = p^{-4}\, dp\, d\omega_2 . \tag{6}$$

Let K_0 be a convex body which contains O in its interior, and let $p(\omega_2)$ be the support function of K_0 with respect to O. If we set $F(K_0 \cap H) = 0$ if $K_0 \cap H \neq 0$ and $F(K_0 \cap H) = 1$ if $K_0 \cap H = 0$, (2) reduces to

$$I(O) = \int_{K_0 \cap H = 0} dH = \frac{1}{3} \int p^{-3}\, d\omega_2 \tag{7}$$

where the last integral is extended over the whole 2-sphere. If O is an affine invariant point of K_0 (for instance, its center of gravity), (7) gives an affine invariant for convex bodies (with respect to unimodular affinities). The minimum of I with respect to O is also an affine invariant which we shall represent by I_m.

By comparing I_m with the volume V and the affine area F_a of K_0 the following theorem can be shown: *Between the unimodular affine invariants I_m, F_a, and V of a convex body the inequalities*

$$I_m V \leq (4\pi/3)^2, \qquad I_m F_a^2 \leq (2^6/3)\pi^2 \tag{8}$$

hold, where the equalities hold only if K is an ellipsoid.

For the analogous relations for the plane see [3]. I do not know if in (8) I_m can be replaced by the invariant $I(O)$ corresponding to the center of gravity of K_0.

3. The group of motions in a space of constant curvature. The best known case is that in which $E = S_n$ is an n-dimensional space of constant curvature k and G is the group of motions in it. In this case the invariant element of volume dx in G is well known. If (P_0, e_0^i) $(i = 1, 2, 3, \cdots, n)$ denotes a fixed n-frame (i.e., a point P_0 and n unit mutually orthogonal vectors with the origin at P_0), any motion x can be determined by the n-frame $(P = xP_0, e^i = xe_0^i)$. Let dP be the element of volume in S_n at P and let $d\omega_{n-1}$ be the element of area on the unit euclidean $(n-1)$-sphere corresponding to the direction of e^1; let $d\omega_{n-2}$ be the element of area on the unit euclidean $(n-2)$-sphere orthogonal to e^1 corresponding to the direction of e^2 and so forth. Then dx can be written

$$dx = [dP\, d\omega_{n-1}\, d\omega_{n-2} \cdots d\omega_1]. \tag{9}$$

Let C_p be a p-dimensional variety $(p \leqq n)$ of finite p-dimensional area A_p and C_q a q-dimensional variety $(q \leqq n)$ of q-dimensional area A_q. Assuming $p + q \geqq n$, let $A_{p+q-n}(C_p \cap xC_q)$ be the $(p + q - n)$-dimensional area of $C_p \cap xC_q$. Then the formula

$$\int_G A_{p+q-n}(C_p \cap xC_q)\, dx = \frac{\omega_{p+q-n}}{\omega_p \omega_q} \omega_1 \omega_2 \cdots \omega_n A_p A_q \tag{10}$$

holds, where ω_i denotes the area of the euclidean unit i-sphere, that is,

$$\omega_i = \frac{2\pi^{(i+1)/2}}{\Gamma((i+1)/2)}. \tag{11}$$

If $p + q = n$, $A_{p+q-n}(C_p \cap xC_q)$ denotes the number of points of the intersection $C_p \cap xC_q$. Notice that (10) is independent of the curvature k of S_n.

If, instead of C_q, we consider a q-dimensional linear subspace L_q^0 of S_n, and denote by $A_{p+q-n}(C_p \cap L_q)$ the area of the intersection of C_p with $L_q = xL_q^0$, we obtain

$$\int_{G/g} A_{p+q-n}(C_p \cap L_q)\, dL_q = \frac{\omega_{p+q-n}\,\omega_{n-q}}{\omega_p} A_p \tag{12}$$

where g is the group of motions which leaves invariant L_q^0 and dL_q is the invariant element of volume in the homogeneous space G/g normalized in such a way that the measure of the L_q which cut the unit $(n-q)$-sphere in S_n be its area (depending upon the curvature k of the space).

We shall give two applications:

a) Let S_n be the euclidean n-sphere; G is then compact and according to (9) its total volume will be $\int_G dx = \omega_1\omega_2 \cdots \omega_n$. In this case the mean value of $A_{p+q-n}(C_p \cap xC_q)$ will be $\omega_{p+q-n}\omega_p^{-1}\omega_q^{-1}A_pA_q$. Consequently we have: *Given on the euclidean n-sphere two varieties C_p, C_q of dimensions p, q $(p + q \geqq n)$ and finite areas A_p, A_q, there exists a motion x such that the area of the intersection $C_p \cap xC_q$ is equal to or greater than $\omega_{p+q-n}\omega_p^{-1}\omega_q^{-1}A_pA_q$.*

b) We shall now give an application to the elementary non-euclidean geometry.

Let T be a tetrahedron in the 3-dimensional space of constant curvature k and let L_2 be the planes of this space. One can show that

$$\int_{T \cap L_2 \neq 0} dL_2 = \frac{1}{\pi} \left(\sum_1^6 (\pi - \alpha_i) l_i + 2kV \right) \tag{13}$$

where l_i are the lengths of the edges of T and α_i the corresponding dihedral angles; V is the volume of T.

On the other hand (12) applied to the edges of T gives

$$\int_{G/\theta} N(T \cap L_2)\, dL_2 = 2 \sum_1^6 l_i \tag{14}$$

where $N(T \cap L_2)$ denotes the number of edges which are intersected by L_2 and therefore is either $N = 3$ or $N = 4$. From (13) and (14) we can evaluate the measures of the sets of planes L_2 corresponding to $N = 3$ and $N = 4$. These measures being non-negative we get the inequalities (for the euclidean case see [5])

$$2 \sum_1^6 \alpha_i l_i - 4kV \leq \pi \sum_1^6 l_i \leq 3 \sum_1^6 \alpha_i l_i - 6kV$$

which for $k = 1$, $k = -1$ gives the following inequalities for the volume V of a tetrahedron in non-euclidean geometry

$$\frac{1}{4} \sum_1^6 (2\alpha_i - \pi) l_i \leq V \leq \frac{1}{6} \sum_1^6 (3\alpha_i - \pi) l_i \quad \text{for the elliptic space,}$$

$$\frac{1}{6} \sum_1^6 (\pi - 3\alpha_i) l_i \leq V \leq \frac{1}{4} \sum_1^6 (\pi - 2\alpha_i) l_i \quad \text{for the hyperbolic space.}$$

These inequalities may have some interest because, as is known, V cannot be expressed in terms of elementary functions of l_i and α_i.

4. A definition of p-dimensional measure of a set of points in euclidean n-space. Let E be now the euclidean n-dimensional space E_n. The methods of Integral Geometry can be used in order to give a definition of area for p-dimensional surfaces (see Maak [3], Federer [2], and for a comparative analysis Nöbeling [4]). The idea of the method is as follows. The formulas (10) and (12) hold for varieties which have a well-defined p- and q-dimensional area in the classical sense. For more general varieties the same formulas (10), (12) can be taken as a definition for A_p (taking for C_q a variety with A_q well-defined), *provided the integrals on the left-hand sides exist.* The problem is therefore to find the conditions of regularity which C_p must satisfy in order that the integrals (10) or (12) exist.

We want to give an example.

Let C be a set of points in E_n and let dP_i be the element of volume in E_n at the point P_i. Let N_s be the number of common points of C with a unit $(n-1)$-

spheres whose centers are the points $P_1, P_2, P_3, \cdots, P_s$. Let us consider the following integrals (in the sense of Lebesgue)

$$(15) \qquad I_s = \int N_s \, dP_1 \, dP_2 \, dP_3 \cdots dP_s \qquad (s = 1, 2, 3, \cdots)$$

extended with respect to each P_i to the whole E_n.

The q-dimensional measure of C can be defined by the formula

$$(16) \qquad m_q(C) = \frac{\omega_q}{2\omega_n^q} I_q.$$

Notice that if I_r is the first integral in the sequence $I_1, I_2, I_3, \cdots$ which has a finite value (may be zero), then $m_q(C) = \infty$ for $q < r$ and $m_q(C) = 0$ for $q > r$. The number r can be taken as the definition for the *dimension* of C.

If C is a q-dimensional variety with tangent q-plane at every point, (16) gives the ordinary q-dimensional area of C (as may be deduced from (10)). The definition (16) may be applied whenever the integrals (15) exist. Following a method used by Nöbeling [4] in similar cases, it is not difficult to prove that the integrals I_s exist if C is an analytic set (or Suslin set).

5. Application to Hermitian spaces. Let $E = P_n$ be now the n-dimensional complex projective space with the homogeneous coordinates $\xi_0, \xi_1, \cdots, \xi_n$ and let G be the group of linear transformations which leaves invariant the Hermitian form $(\xi\bar{\xi}) = \sum \xi_i \bar{\xi}_i$.

If we normalise the coordinates ξ_i such that $(\xi\bar{\xi}) = 1$, every variety C_p of complex dimension p possesses an invariant integral of degree $2p$, namely $\Omega^p = (\sum [d\xi_i d\bar{\xi}_i])^p$ (see Cartan [1]). Let us put

$$(17) \qquad J_p(C_p) = \frac{p!}{(2\pi i)^p} \int_{C_p} \Omega^p.$$

It is well known that if C_p is an algebraic variety of dimension p, $J_p(C_p)$ coincides with its order.

If C_p is an analytic variety ("synectic" according to Study, i.e., defined by complex analytic relations) the methods of Integral Geometry give a simple interpretation of the invariant J_p. Let L_{n-p}^0 be a linear subspace of dimension $n - p$ and put $L_{n-p} = xL_{n-p}^0$. If g is the subgroup of G which leaves L_{n-p}^0 invariant, and dL_{n-p} means the invariant element of volume in the homogeneous space G/g normalised in such a way that the total volume of G/g is equal 1, the formula

$$(18) \qquad \int_{G/g} N(C_p \cap L_{n-p}) \, dL_{n-p} = J_p(C_p)$$

holds, where $N(C_p \cap L_{n-p})$ denotes the number of points of intersection of C_p with L_{n-p}.

A more general formula, assuming $p + q \geq n$, is the following

$$\int_{G/g} J_{p+q-n}(C_p \cap L_q)\, dL_q = J_p(C_p) \tag{19}$$

which coincides with (18) for $q = n - p$.

If dx is the element of volume in G normalized in such a way that the total volume of G is equal 1, given two analytic varieties C_p, C_q we also have

$$\int_G J_{p+q-n}(C_p \cap xC_q)\, dx = J_p(C_p)J_q(C_q) \tag{20}$$

which may be considered as the generalization to analytic varieties of the theorem of Bezout.

For $p + q = n$, the foregoing formula (20) is a particular case of a much more general result of de Rham [6].

6. Integral geometry in Riemannian spaces. The Integral Geometry in an n-dimensional Riemannian space R_n presents a different aspect. Here we do not have, in general, a group of transformations G. However, if we take as geometrical elements the geodesic curves Γ of R_n, it is possible to consider integrals analogous to (2), though conceptually different, and to deduce from them geometrical consequences.

Let $ds^2 = g_{ij}\, du^i\, du^j$ be the metric in R_n and let us set $\varphi = (g_{ij}\dot{u}^i\dot{u}^j)^{1/2}$ and $p_i = \partial\varphi/\partial\dot{u}^i$. The exterior differential form $d\Gamma = (\sum[dp_i\, du^i])^{n-1}$ of degree $2(n - 1)$ is invariant under displacements of the elements u^i, p_i on the respective geodesic. Therefore we can define the "measure" of a set of geodesic curves as the integral of $d\Gamma$ extended over the set.

Let us consider a bounded region D_0 in R_n and let different arcs of geodesic contained in D_0 be taken as different geodesic lines. Let us consider a geodesic Γ which intersects an $(n - 1)$-dimensional variety C_{n-1} contained in D_0 at the point P. Let $d\sigma$ denote the element of $(n - 1)$-dimensional area on C_{n-1} at P. If $d\omega_{n-1}$ denotes the element of area on the unit euclidean $(n - 1)$-sphere corresponding to the direction of the tangent to Γ at P and θ denotes the angle between Γ and the normal to C_{n-1} at P, the differential form $d\Gamma$ may be written in the form $d\Gamma = |\cos\theta|\, [d\omega_{n-1}d\sigma]$. If C_{n-1} has a finite $(n - 1)$-dimensional area A_{n-1} and $N(C_{n-1} \cap \Gamma)$ denotes the number of intersection points of C_{n-1} and Γ, from the last form for $d\Gamma$ it follows that

$$\int_{D_0} N(C_{n-1} \cap \Gamma)\, d\Gamma = \frac{\omega_{n-2}}{n-1} A_{n-1} \tag{21}$$

where the integral is extended over all geodesics of D_0.

If dt denotes the element of arc on Γ and dP is the element of volume in R_n at P, clearly we have $[d\Gamma\, dt] = [dP\, d\omega_{n-1}]$. From this relation if we consider all arc elements (Γ, t) with the origin within a given region D (contained in D_0)

of finite volume $V(D)$ and call $L(D \cap \Gamma)$ the length of the arc of Γ which lies within D, we obtain

$$\int_{D_0} L(D \cap \Gamma)\, d\Gamma = \frac{1}{2}\omega_{n-1} V(D). \tag{22}$$

Some consequences of the formulas (21) and (22) for the case $n = 2$ have been given in [8]. They have particular interest for the Riemannian spaces of finite volume whose geodesic lines are all closed curves of finite length (for $n \geq 3$ it seems, however, not to be known if such spaces, other than spheres, exist). For instance, one can easily show: *If the geodesic lines of a Riemannian space R_n of finite volume V are all closed curves of constant length L and there exists in R_n an $(n-1)$-dimensional variety of area A_{n-1} which intersects all the geodesic curves, the inequality*

$$LA_{n-1} \geq \frac{(n-1)\omega_{n-1}}{2\omega_{n-2}} V$$

holds (equality for the elliptic space).

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