ON THE MEASURE OF LINE SEGMENTS ENTIRELY CONTAINED IN A CONVEX BODY

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Dedicated to Leopoldo Nachbin with admiration and friendship

Let \( K \) be a convex body in the \( n \)-dimensional euclidean space \( \mathbb{R}^n \). We consider the measure \( M_0(l) \), in the sense of the integral geometry (i.e. invariant under the group of translations and rotations of \( \mathbb{R}^n \) [6, Chap. 15]), of the set of non-oriented line segments of length \( l \), which are entirely contained in \( K \). This measure is related by (3.4) with the integrals \( I_m \) for the power of the chords of \( K \). These relations allow to obtain some inequalities, like (3.6), (3.7) and (3.8) for \( M_0(l) \). Next we relate \( M_0(l) \) with the function \( \Omega(l) \) introduced by Enns and Ehlers [3], and prove a conjecture of these authors about the maximum of the average of the random straight line path through \( K \). Finally, for \( n = 2 \), \( M_0(l) \) is shown to be related by (5.6) with the associated function to \( K \) introduced by W. Pohl [5]. Some representation formulas, like (3.9), (3.10) and (5.14) may be of independent interest.

1. Integrals for the Power of the Chords of a Convex Body

Let \( K \) be a convex body in the \( n \)-dimensional euclidean space \( \mathbb{R}^n \). Let \( dG \) be the density for lines \( G \) in \( \mathbb{R}^n \) in the sense of integral geometry [6, Chap. 12] and let \( \sigma \) denote the length of the chord \( G \cap K \). The chord power integrals

\[
I_m = \int_{G \cap K \neq \emptyset} \sigma^m \, dG \quad (m \geq 0),
\]

have been well studied [6, p. 237]. If \( dP_1, dP_2 \) denote the elements of volume of \( \mathbb{R}^n \) at the points \( P_1, P_2 \in K \) and \( r \) denotes the distance between \( P_1 \) and \( P_2 \), the integrals

\[
J_m = \int_{P_1, P_2 \in K} r^m \, dP_1 \wedge dP_2 \quad (m \geq - (n - 1)),
\]
have also been considered and it is known that the relation

\(2I_m = m(m-1)J_{m-n-1}\)

holds good for \(m > 1\) [6, p. 238].

For the cases \(m = 0, 1\) and \(m = n + 1\) we have the simple formulas

\[ I_0 = \frac{1}{2} O_{n-2} F, \quad I_1 = \frac{1}{2} O_{n-1} V, \quad I_{n+1} = \left(\frac{1}{2} n(n+1)\right) V^2, \]

where \(F\) is the surface area of \(K\) and \(V\) its volume.

We want to calculate \(I_m\) for the sphere \(S_r\) of radius \(r\) in \(\mathbb{R}^n\). To this end, recalling that \(dG = d\sigma_{n-1} \wedge dO_{n-1}\) [6, (12.39)] where \(d\sigma_{n-1}\) is the area element of an hyperplane orthogonal to \(G\) at its intersection point with \(G\) and \(dO_{n-1}\) is the area element of the unit sphere at the end point of the unit vector parallel to \(G\), we can write \(dG = \rho^{n-2} dO_{n-2} \wedge d\rho \wedge dO_{n-1}\) and therefore we have (\(\rho\) being the distance from the center of the sphere to \(G\))

\[ I_m = 2^{m-1} O_{n-1} O_{n-2} \int_0^r (r^2 - \rho^2)^{m/2} \rho^{n-2} d\rho \]

\[ = 2^{m-2} O_{n-1} O_{n-2} \rho^{m+n-1} B\left(\frac{1}{2}(n-1), \frac{1}{2}(m+2)\right), \]

where \(B(p, q) = \Gamma(p) \Gamma(q)/\Gamma(p + q)\) is the Beta function and \(O_h\) means the surface area of the \(h\)-dimensional unit sphere, i.e.

\[ O_h = \frac{2\pi^{(h+1)/2}}{\Gamma\left(\frac{1}{2}(h+1)\right)}. \]

Therefore we have

\[ I_m(S_r) = \frac{2^{m-1} r^{m+n-1} \pi^{n-1/2} m \Gamma\left(\frac{1}{2} m\right)}{\Gamma\left(\frac{1}{2} n\right) \Gamma\left(\frac{1}{2} (m+n+1)\right)}. \]

2. Inequalities of Hadwiger, Carleman and Blaschke for the Chord Integrals \(I_m\)

The chord integrals \(I_m\) for convex bodies in \(\mathbb{R}^n\) satisfy certain inequalities. One of them is due to H. Hadwiger [4]:

\[
2I_m = m(m-1)J_{m-n-1}
\]

holds good for \(m > 1\) [6, p. 238].

For the cases \(m = 0, 1\) and \(m = n + 1\) we have the simple formulas

\[ I_0 = \frac{1}{2} O_{n-2} F, \quad I_1 = \frac{1}{2} O_{n-1} V, \quad I_{n+1} = \left(\frac{1}{2} n(n+1)\right) V^2, \]

where \(F\) is the surface area of \(K\) and \(V\) its volume.

We want to calculate \(I_m\) for the sphere \(S_r\) of radius \(r\) in \(\mathbb{R}^n\). To this end, recalling that \(dG = d\sigma_{n-1} \wedge dO_{n-1}\) [6, (12.39)] where \(d\sigma_{n-1}\) is the area element of an hyperplane orthogonal to \(G\) at its intersection point with \(G\) and \(dO_{n-1}\) is the area element of the unit sphere at the end point of the unit vector parallel to \(G\), we can write \(dG = \rho^{n-2} dO_{n-2} \wedge d\rho \wedge dO_{n-1}\) and therefore we have (\(\rho\) being the distance from the center of the sphere to \(G\))

\[ I_m = 2^{m-1} O_{n-1} O_{n-2} \int_0^r (r^2 - \rho^2)^{m/2} \rho^{n-2} d\rho \]

\[ = 2^{m-2} O_{n-1} O_{n-2} \rho^{m+n-1} B\left(\frac{1}{2}(n-1), \frac{1}{2}(m+2)\right), \]

where \(B(p, q) = \Gamma(p) \Gamma(q)/\Gamma(p + q)\) is the Beta function and \(O_h\) means the surface area of the \(h\)-dimensional unit sphere, i.e.

\[ O_h = \frac{2\pi^{(h+1)/2}}{\Gamma\left(\frac{1}{2}(h+1)\right)}. \]

Therefore we have

\[ I_m(S_r) = \frac{2^{m-1} r^{m+n-1} \pi^{n-1/2} m \Gamma\left(\frac{1}{2} m\right)}{\Gamma\left(\frac{1}{2} n\right) \Gamma\left(\frac{1}{2} (m+n+1)\right)}. \]
(2.1) \[ 2I_{n-1} \leq n \left( \frac{\kappa_{n-2}}{n\kappa_{n-1}} \right)^{1/n} F V^{1-1/n} \quad (n > 2), \]

where \( \kappa_n \) denotes the volume of the \( n \)-dimensional unit ball, i.e.

(2.2) \[ \kappa_n = \frac{O_{n-1}}{h} = \frac{2\pi^{n/2}}{h \Gamma\left(\frac{1}{2}h\right)}, \]

and \( F \) and \( V \) denote the surface area and the volume of \( K \) respectively.

Taking into account the isoperimetric inequality

(2.3) \[ nk_n^{1/n} V^{1-1/n} \leq F, \]

inequality (2.1) gives

(2.4) \[ 4I_{n-1} \leq F^2 \quad (n > 2). \]

In (2.1) and (2.4) the equality sign holds only for the sphere.

In [2] T. Carleman proved that in the plane, \( n = 2, \) \( J_{-1} = \int r^{-1} dP_1 \wedge dP_2 = I_2 \) has its maximum for the circle (for a given surface area) and pointed out that the same proof may be extended to showing that for convex bodies in \( R^n, \) the integrals \( I_m \) for \( m = 2, 3, \ldots, n \) have a maximum for the sphere for a given volume \( V. \) Thus, taking into account (1.7) and (2.2) we have the following set of inequalities:

(2.5) \[ I_m^n \leq 2^{mn-m-n+1} \pi^{(n^2/2)-mn/2} n^{m+n-1} (\Gamma\left(\frac{1}{2} n\right))^{m-1} \left( \frac{\Gamma\left(\frac{1}{2}m + 1\right)}{\Gamma\left(\frac{1}{2}(m + n + 1)\right)} \right)^n V^{m+n-1}, \]

for \( m = 2, 3, \ldots, n. \)

In [1], W. Blaschke proved that in the plane \( (n = 2) \) and for a given area \( F, \) the integrals \( I_m \) \( (m \geq 4) \) have its minimum for the circle. The proof is also easily extendible to \( R^n, \) so that, taking (1.7) and (2.2) into account, we can write the new set of inequalities (for \( R^n \))

(2.6) \[ I_m^n \geq 2^{mn-m-n+1} \pi^{(n^2/2)-mn/2} n^{m+n-1} (\Gamma\left(\frac{1}{2} n\right))^{m-1} \left( \frac{\Gamma\left(\frac{1}{2}m + 1\right)}{\Gamma\left(\frac{1}{2}(m + n + 1)\right)} \right)^n V^{m+n-1}, \]

for \( m \geq n + 2. \) In (2.5) and (2.6) the equality sign holds only for the sphere.
3. The Measure $M_0(l)$ of the Line Segments of Length $l$ Entirely Contained in a Convex Body $K$ in $\mathbb{R}^n$

A line segment $S$ of given length $l$ in $\mathbb{R}^n$ can be determined either by the line $G$ which contains the segment and the abscissa $t$ of the origin $P$ of $S$ on $G$, or by $P$ and the point on the unit sphere $O_{n-1}$ given by the direction of $S$. The kinematic density for sets of line segments of length $l$ (invariant under motions in $\mathbb{R}^n$) is [6, p. 338]:

\begin{equation}
(3.1) \quad dS = dG \wedge dt = dP \wedge dO_{n-1}.
\end{equation}

Using $dS = dG \wedge dt$ we get that the measure of the set of line segments $S$ entirely contained in $K$ is

\begin{equation}
(3.2) \quad M_0(l) = \int_{a<l} (\sigma - l) dG.
\end{equation}

If $P_1, P_2$ are two points of $K$ at a distance $l$, we have $dP_1 \wedge dP_2 = l^{n-1} dO_{n-1} \wedge dl \wedge dP_1$ (up to the sign) and therefore, since we consider the measure of non-oriented segments, we have

\begin{equation}
(3.3) \quad \int_{P_1, P_2 \in K} l^m dP_1 \wedge dP_2 = 2 \int_0 l^{m+n-1} M_0(l) dl.
\end{equation}

As a consequence of (1.3) and (3.3) we have

\begin{equation}
(3.4) \quad I_m = m(m-1)J_{m-n-1} = m(m-1) \int_0 l^{m-2} M_0(l) dl,
\end{equation}

which holds for $m \geq 2$. In particular, for $m = 2$ we have

\begin{equation}
(3.5) \quad I_2 = 2 \int_0 M_0(l) dl,
\end{equation}

and the first inequality (2.5) gives
where the equality sign holds only for the sphere.

For instance, for convex sets \( K \) in the plane, \( n = 2 \), we have

\[
\int_0^{\text{Diam}(K)} M_0(l) \, dl \leq \frac{8}{3\sqrt{\pi}} F^{3/2},
\]

where \( F \) is the surface area of \( K \).

Taking into account the isoperimetric inequality \( 4\pi F \leq L^2 \) and the inequality of Bieberbach \( F \leq \frac{1}{4} \pi D^2 \), where \( D = \text{diam}(K) \), we get the following inequalities (for convex sets in the plane):

\[
\int_0^D M_0(l) \, dl \leq \frac{L^3}{3\pi^2}, \quad \int_0^D M_0(l) \, dl \leq \frac{1}{3} \pi D^3,
\]

with the equality sign always only for the circle.

From (3.4) we deduce that for every polynomial function of the form \( f = a_2 \sigma^2 + \cdots + a_h \sigma^h \) we have

\[
\int_{\partial\mathcal{K} \neq \emptyset} f(\sigma) \, dG = \int_0^D f''(\sigma) M_0(\sigma) \, d\sigma.
\]

By Weierstrass approximation theorem, this equality holds for every function \( f(\sigma) \) having continuous derivatives \( f''(\sigma) \) with the conditions \( f(0) = f'(0) = 0 \).

Integrating by parts the right side of (3.9) we have the following relationship

\[
\int_{\partial\mathcal{K} \neq \emptyset} f(\sigma) \, dG = - \int_0^D f'(\sigma) M'_0(\sigma) \, d\sigma,
\]
for every function $f(\sigma)$ having continuous derivative $f'(\sigma)$ and satisfying the condition $f(0) = f'(0) = 0$.

4. The Invariants $\Omega(l)$ of Enns–Ehlers

Denote by $K(l, \omega)$ the translate of $K$ by a distance $l$ in the direction $\omega$. Enns and Ehlers [3] define $\Omega(l)$ to be the volume of $K \cap K(l, \omega)$ uniformly averaged over all directions and normalized such that $\Omega(0) = 1$. If $\sigma$ denotes the length of the chord $G \cap K$, the volume of $K \cap K(l, \omega)$ is precisely $\int_{\sigma > l} (\sigma - l) \, d\sigma_{n-1}$, where $d\sigma_{n-1}$ denotes the area element on the hyperplane orthogonal to the line $G$ which has the direction $\omega$. Therefore, since $dG = d\sigma_{n-1} \wedge d\Omega_{n-1}$, where $d\Omega_{n-1}$ denotes the area element on the unit $(n - 1)$-sphere corresponding to the direction $\omega$, we have

$$\int_{\sigma > l} (\sigma - l) \, d\sigma_{n-1} \wedge d\Omega_{n-1} = \frac{2}{O_{n-1} V} \int_{\sigma > l} (\sigma - l) \, dG$$

and thus, according to (3.2),

$$\Omega(l) = \frac{2}{O_{n-1} V} \int_{\sigma > l} (\sigma - l) \, dG = \frac{2}{O_{n-1} V} \int_{\sigma > l} (\sigma - l) \, dG$$

(4.1)

and therefore, according to (3.2),

$$\Omega(l) = \frac{2}{O_{n-1} V} M_0(l).$$

(4.2)

Therefore, (3.4) gives

$$I_m = \frac{1}{2} m (m - 1) O_{n-1} V \int_0^D l^{m-2} \Omega(l) \, dl.$$  

(4.3)

For instance, if $m = n + 1$, taking (1.4) into account, we have

$$\int_0^D l^{n-2} \Omega(l) \, dl = \frac{V}{O_{n-1}},$$

(4.4)

according to a result of Enns and Ehlers [3, (8)].

If a 'random secant' is defined by a point in the interior of $K$ and by a direction (the point and direction have independent uniform distribution),
the k-th moment of a random secant is (using (3.1))

\[ (4.5) \quad E(\sigma^k) = \frac{2}{O_{n-1} V} \int \sigma^k \, dP \wedge dO_{n-1} = \frac{2}{O_{n-1} V} \int \sigma^{k+1} \, dG \]

\[ = \frac{2k(k + 1)}{O_{n-1} V} \int_0^D \sigma^{k-1} M_0(\sigma) \, d\sigma. \]

Thus, according to (3.4) we have

\[ E(\sigma^k) = \frac{2}{O_{n-1} V} I_{k+1}, \]

and the inequalities (2.5) give

\[ (4.6) \quad E(\sigma^k) \leq \frac{2^{k-k/n} n^{(k/n)+1} \Gamma(\frac{1}{2} n) \Gamma^2(\frac{1}{2} n) \Gamma(\frac{1}{2} (k + 3))}{\pi^{(k+1)/2} \Gamma^2(\frac{1}{2} (k + n + 2))} V^{k/n}, \]

which holds for k = 1, 2, \ldots, n - 1 and the equality sign holds only for the sphere. For the sphere of radius r we have \( V = (2\pi^{n/2}/n\Gamma(\frac{1}{2} n))r^n \) and therefore

\[ (4.7) \quad E(\sigma^k) = \frac{2^k n \Gamma(\frac{1}{2} n) \Gamma^2(\frac{1}{2} (k + 3))}{\pi^{(k+1)/2} \Gamma^2(\frac{1}{2} (k + n + 2))} r^k, \]

as is well-known (Enns–Ehlers [3]).

In particular (4.6) implies that of all n-dimensional convex bodies K of volume V, \( E(\sigma) \) is maximized for the n-sphere. This proves a conjecture of Enns and Ehlers [3].

The inequalities (2.6) can be written

\[ (4.8) \quad E(\sigma^k) \geq \frac{2^{k-k/n} n^{1+k/n} \Gamma(\frac{1}{2} n) \Gamma^2(\frac{1}{2} n) \Gamma(\frac{1}{2} (k + 3))}{\pi^{(k+1)/2} \Gamma^2(\frac{1}{2} (k + n + 2))} V^{k/n}, \]

valid for k = n + 1, n + 2, \ldots. The equality sign holds only for the sphere.

For the plane, \( n = 2 \), if \( F \) denotes the area of \( K \), we have

\[ (4.9) \quad E(\sigma) \leq \frac{8F^{1/2}}{3\pi^{3/2}}, \quad E(\sigma^2) = \frac{3}{\pi} F, \]
and therefore, of all the convex sets of area $F$, the variance

$$E(\sigma^2) - (E(\sigma))^2 \geq \frac{27\pi^2 - 64}{9\pi^3} F$$

is minimized for the circle (as conjectured by Enns–Ehlers [3]). The conjecture that the variance is also minimized for the sphere if $n > 2$ remains open.

5. The Associated Functions $A(\sigma)$ of W. Pohl

In this section we consider only the case of the plane, $n = 2$. In each line $G$ we choose a point $X(x, y)$ and the unit vector $e(\cos \theta, \sin \theta)$ corresponding to its direction. Consider the differential form $\omega = \langle dX, e \rangle = \cos \theta \cdot dx + \sin \theta \cdot dy$. Then we have $d\omega = -\sin \theta \cdot d\theta \wedge dx + \cos \theta \cdot d\theta \wedge dy = dG$ (according to [6, (3.11)]). W. Pohl [5] defines the associated function $A(\sigma)$ to the convex curve $\partial K$ by

$$A(\sigma) = \int_{\partial K} \cos \alpha \, ds,$$

where the integral of $\omega$ extends to the non-oriented lines $(X, e), X \in \partial K$, that determine on the convex set $K$ a chord of length $\sigma$ and in the last integral $\alpha$ denotes the angle between the tangent to $\partial K$ and $G$ at the point $X$ corresponding to the element of the arc $ds$.

A simple geometric description of $A(\sigma)$, at least for small values of $\sigma$, is the following [5]: Let $\partial K_0$ be the curve envelope of the chords of $K$ of length $\sigma$. Then $A(\sigma)$ is length of $\partial K_0$. For instance, for a circle of diameter $D$ we have

$$A(\sigma) = \pi(D^2 - \sigma^2)^{1/2}.$$ 

Notice that our $A(\sigma)$ is one half of that of Pohl, which considers oriented lines.

Let $M_1(l)$ be the measure of the set of non-oriented line segments of length $l$ such that one end point is inside $K$ and one outside $K$. Then we have [5]
(5.3) \[ M_1(l) = 2 \int_0^l A(\sigma) \, d\sigma. \]

On the other side, using the kinematic density \( dS = dG \wedge dt \), we have

(5.4) \[ M_1(l) = 2 \int_{\sigma > l} l \, dG + 2 \int_{\sigma < l} \sigma \, dG \]

and by virtue of (3.2) we get

(5.5) \[ M_0 + \frac{1}{2}M_1 = \pi F, \]

where \( F \) is the surface area of \( K \). From (5.3) and (5.5) we have

(5.6) \[ M_0 = \pi F - \int_0^l A(\sigma) \, d\sigma, \]

and

(5.7) \[ A(\sigma) = -M'_0(\sigma). \]

The relation (5.6) can be applied to compute the measure \( M_0(l) \) of non-oriented line segments of length \( l \leq D \) entirely contained in a circle of diameter \( D \). Namely, from (5.2) we have

(5.8) \[ M_0(l) = \pi F - \frac{1}{2} \int_0^l (D^2 - \sigma^2)^{1/2} \, d\sigma \]

\[ = \frac{1}{4} \pi \left( \pi D^2 - 2l(D^2 - l^2)^{1/2} - 2D^2 \arcsin \left( \frac{l}{D} \right) \right), \]

as is well known [6, p. 90].

Integrating by parts in (3.4) and taking into account (5.7), we get (for convex sets in the plane and \( m \geq 1 \))
\[ I_m = m \int_0^D \sigma^{m-1} A(\sigma) \, d\sigma, \]

where \(D\) is the diameter of \(K\). This expression for the chord integrals \(I_m\) (for convex sets on the plane) is due to Pohl [5]. For \(m = 1\) we have

\[ \int_0^D A(\sigma) \, d\sigma = \pi F. \]

For \(m = 2\), according to (2.5) we get the inequality

\[ \int_0^D \sigma A(\sigma) \, d\sigma \leq \frac{8}{3\sqrt{\pi}} F^{3/2}. \]

For \(m = 3\) we have

\[ \int_0^D \sigma^2 A(\sigma) \, d\sigma = F^2, \]

and for \(m > 3\),

\[ \int_0^D \sigma^{m-1} A(\sigma) \, d\sigma \geq \frac{2^{m-1} \pi^{1-m/2} \Gamma(\frac{1}{2} m)}{\Gamma(\frac{1}{2} (m + 3))} \Gamma^{m+1/2}. \]

In (5.11) and (5.13) the equality sign holds only for the circle.

From (3.10) and (5.7) we have

\[ \int \int_{\sigma \cap \kappa \neq \emptyset} f(\sigma) \, dG = \int_0^D f'(\sigma) A(\sigma) \, d\sigma, \]

which holds for every function \(f(\sigma)\) having a continuous derivative \(f'(\sigma)\) and satisfying the conditions \(f(0) = f'(0) = 0\).
The relation between the invariant $\Omega(\sigma)$ of Enns–Ehlers and the associated function $A(\sigma)$ of Pohl, according to (4.2) and (5.6) is

$$A(\sigma) = -\pi F\Omega'(\sigma).$$

References