ON THE MEAN CURVATURES OF A FLATTENED CONVEX BODY

L. A. SANTALÓ

Let E^n be the n-dimensional euclidean space and let L^r be a linear subspace of E^n. A convex body K^r contained in L^r can be considered as a *flattened convex body* of E^n. As a convex body of L^r, K^r possesses the mean curvatures

$$M_q^r \, (0 \leq q \leq r-1)$$

defined by (1). As a flattened convex body of E^n, K^r possesses the mean curvatures

$$M_q^n \, (0 \leq q \leq n-1)$$

defined as the limit of the mean curvatures $M_q^n(\varepsilon)$ of the convex body $K^r(\varepsilon)$ parallel exterior to K^r at the distance ε, as $\varepsilon \to 0$.

The purpose of the present note is to prove the formulae (13), (14), (15) which relate the mean curvatures M_q^r and M_q^n. As a consequence, we complete an integral-geometric result of HERGLOTZ and PETKANTSCHIN [formula (16)].

1. Let K^r be a convex body contained in a linear subspace L^r of E^n. The boundary ∂K^r is an $(r-1)$-dimensional variety of L^r which is assumed to be twice differentiable.

If ϱ_i $(i = 1, 2, \ldots, r-1)$ denote the principal radii of curvature of ∂K^r at a point P', the q-th mean curvature of K^r (as a convex body of L^r) is defined by

$$M_q^r = \frac{1}{\binom{r-1}{q}} \int_{\partial K^r} \left\{ \frac{1}{\varrho_1}, \ldots, \frac{1}{\varrho_q} \right\} d\sigma_{r-1} \tag{1}$$

where the brackets $\{\}$ denote the q-th elementary symmetric function formed by the principal curvatures $1/\varrho_i$ and $d\sigma_{r-1}$ is the element of area of ∂K^r at P'.

As particular cases we have:

$M_0^r = \sigma_{r-1}$ = area of ∂K^r,

$M_{r-1}^r = O_{r-1}$ = area of the $(r-1)$-dimensional unit sphere, *i. e.*

$$O_{r-1} = \frac{2\pi^{r/2}}{\Gamma(r/2)} \tag{2}$$

For instance, if $r=2$, K^2 is a plane convex figure and we have: $M_0^2=\sigma_1=$ length of the boundary of K^2; $M_1^2=2\pi$. For $r=3$, if K^3 is a convex body in ordinary space, we have: $M_0^3=\sigma_2=$ surface area of ∂K^3; $M_1^3=$ integrated mean curvature of ∂K^3; $M_2^3=4\pi$. For $r=1$, M_0^1 is meaningless; however in this case we have:

$$M_{r-1}^r=O_{r-1}=O_0=2$$

and consequently we will allways take $M_0^1=2$.

We now consider K^r as a flattened convex body of E^n. In order to define its mean curvatures

$$M_q^n \quad (q=1,2,\ldots,n-1)$$

we consider first the mean curvatures of the convex body $K^r(\varepsilon)$ parallel to K^r at a distance ε (*i. e.* the set of points of E^n whose distance to K^r is $\leq \varepsilon$) and then pass to the limit as $\varepsilon \to 0$.

The boundary $\partial K^r(\varepsilon)$ is a twice differentiable hypersurface of E^n with a well defined normal at each point P; let P' be the intersection point of K^r (considered as a convex body of L^r) we will say that P belongs to the region (A) of $\partial K^r(\varepsilon)$; if P' is a point of ∂K^r we will say that P belongs to the region (B) of $\partial K^r(\varepsilon)$.

At the points of the region (A), the element of area of $\partial K^r(\varepsilon)$ is equal to $\varepsilon^{n-r-1}\,dO_{n-r-1}\,d\sigma_r$ where $dO_{n-r-1}=$ area element of the unit $(n-r-1)$-dimensional sphere and $d\sigma_r=$ volume element of K^r. At the points of the region (B), the element of area of $\partial K^r(\varepsilon)$ is equal to $\varepsilon^{n-r}\,dO_{n-r}\,d\sigma_{r-1}$ where $d\sigma_{r-1}=$ element of area of ∂K^r. Consequently, the q-th mean curvature of $K^r(\varepsilon)$ is given by

$$M_q^n(\varepsilon)=\frac{1}{\binom{n-1}{q}}\Big[\int_{K^r}\Big\{\frac{1}{R_1},\ldots,\frac{1}{R_q}\Big\}\varepsilon^{n-r-1}\,dO_{n-r-1}\,d\sigma_r + \int_{\partial K^r}\Big\{\frac{1}{R_1},\ldots,\frac{1}{R_q}\Big\}\varepsilon^{n-r}\,dO_{n-r}\,d\sigma_{r-1}\Big] \tag{3}$$

where the principal radii of curvature $R_h=R_h(\varepsilon)$ have the following values:

a) For the points of the region (A) it is clear that

$$\begin{aligned} R_h&=\varepsilon \quad \text{for} \quad h=1,2,3,\ldots,n-r-1\\ R_h&=\infty \quad \text{for} \quad h=n-r,\,n-r+1,\ldots,n-1. \end{aligned} \tag{4}$$

b) In order to find the values of $R_h=R_h(\varepsilon)$ at the points of the region (B), let us consider at each point x of ∂K^r a frame of n orthogonal unit vectors $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ such that $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_{n-r}$ be constant (independent of x) and orthogonal to L^r; $\mathbf{e}_{n-r+1},\ldots,\mathbf{e}_{n-1}$ be the principal tangents to ∂K^r (as a variety of L^r) at the point x, and $\mathbf{e}_n$ be the normal to ∂K^r contained in L^r. The vector equation of $\partial K^r(\varepsilon)$ will be

$$\mathbf{X} = \mathbf{x} - \varepsilon \mathbf{N} \tag{5}$$

where

$$\mathbf{N} = \cos\vartheta\, \mathbf{e}_n + \sum_{h=1}^{n-r} \cos\vartheta_h\, \mathbf{e}_h . \tag{6}$$

For each fixed $\mathbf{x}$, $\mathbf{X}$ will describe a $(n-r)$-sphere and consequently we have

$$R_h = \varepsilon \quad \text{for} \quad h = 1, 2, 3, \ldots, n-r. \tag{7}$$

For $h = n-r+1, \ldots, n-1$, by the equations of Olinde Rodrigues we have

$$d\mathbf{N} \cdot \mathbf{e}_h = -\frac{1}{R_h}\, d\,S_h \tag{8}$$

where $d\,S_h$ denotes the arc element on $\partial K^r(\varepsilon)$ the tangent vector of this arc being parallel to $\mathbf{e}_h$, *i. e.*

$$d\,S_h = d\,\mathbf{X} \cdot \mathbf{e}_h = ds_h - \varepsilon\, d\,\mathbf{N} \cdot \mathbf{e}_h \tag{9}$$

where ds_h is the arc element on ∂K^r tangent to $\mathbf{e}_h$.

From (6), taking into account that the vectors $\mathbf{e}_h$, for $h = 1, 2, \ldots, n-r$, are constant, we have

$$d\mathbf{N} \cdot \mathbf{e}_h = \cos\vartheta\, d\,\mathbf{e}_n \cdot \mathbf{e}_h = -\frac{\cos\vartheta}{\varrho_{h-n+r}}\, ds_h \quad (h = n-r+1, \ldots, n-1) \tag{10}$$

where $\varrho_1, \varrho_2, \ldots, \varrho_{r-1}$ are the principal radii of curvature of ∂K^r. From (8), (9) and (10) we have

$$R_h = \frac{\varrho_{h-n+r}}{\cos\vartheta} + \varepsilon \quad \text{for} \quad h = n-r+1, \ldots, n-1. \tag{11}$$

With the values (4) and (7), (11) we can calculate $M_q^n(\varepsilon)$ and pass to the limit as $\varepsilon \to 0$.

There are three possible cases:

(1) $q \geq n-r$. The first integral in (3) vanishes as $\varepsilon \to 0$ and the second integral reduces to

$$M_q^n = \frac{1}{\binom{n-1}{q}} \int_{\partial K^r} \left\{ \frac{1}{\varrho_1}, \ldots, \frac{1}{\varrho_{q-n+r}} \right\} \cos^{q-n+r}\vartheta\, dO_{n-r}\, d\sigma_{r-1} \tag{12}$$

$$= \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} M_{q-n+r}^r \int \cos^{q-n+r}\vartheta\, dO_{n-r} .$$

The area element of the $(n-r)$-dimensional unit sphere may be written

$$dO_{n-r} = \sin^{n-r-1}\vartheta \sin^{n-r-2}\vartheta_1 \dots \sin\vartheta_{n-r-2}\, d\vartheta\, d\vartheta_1 \dots d\vartheta_{n-r-1}$$

and the integral in (12) must be extended over the half sphere whose pole is the end point of the normal to ∂K^r (contained in L^r). The limits of integration are then

$$0 \leq \vartheta \leq \frac{\pi}{2}, \quad 0 \leq \vartheta_i \leq \pi \ (i = 1, 2, \dots, n-r-2), \quad 0 \leq \vartheta_{n-r-1} \leq 2\pi$$

and we have

$$(13) \qquad M_q^n = \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} M_{q-n+r}^r O_{n-r-1} \int_0^{\pi/2} \cos^{q-n+r}\vartheta \sin^{n-r-1}\vartheta\, d\vartheta$$

$$= \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} M_{q-n+r}^r .$$

(2) $q = n - r - 1$. The second integral in (3) vanishes as $\varepsilon \to 0$ and the first tends to $O_{n-r-1}\sigma_r$. Consequently

$$(14) \qquad M_q^n = \frac{1}{\binom{n-1}{q}} O_{n-r-1}\sigma_r(K^r)$$

where $\sigma_r(K^r)$ denotes the volume of K^r.

(3) $q < n - r - 1$. Both integrals in (3) vanish as $\varepsilon \to 0$, and consequently

$$(15) \qquad M_q^n = 0.$$

2. Examples. For the ordinary space, $n = 3$, we have the following possibilities:

a) $r = 1$. K^r reduces to a segment of length s. The mean curvatures are

$$M_0^3 = 0, \quad M_1^3 = \pi s, \quad M_2^3 = 2\pi M_0^4 = 4\pi.$$

b) $r = 2$. K^r is a plane convex figure; let s be its perimeter and σ its area. We have

$$M_0^3 = 2\sigma, \quad M_1^3 = \frac{\pi}{2} M_0^2 = \frac{\pi}{2} s, \quad M_2^3 = 2 M_1^2 = 4\pi .$$

3. An integral-geometric application. The mean curvatures M_q^n are related with certain invariants H_q^n of K^r which appear in integral geometry. H_q^n $(q=0, 1, \ldots, n-1)$ denotes the measure of the set of q-dimensional linear spaces of E^n which have a common point with K^r. Analogously, if K^r is contained in L^r, then H_q^r $(q=0, 1, 2, \ldots, r-1)$ denotes the measure of the set of q-dimensional linear spaces of L^r which intersect K^r. The invariants M_q^n and H_q^n are related by (see [2], p. 183).

$$H_q^n = \frac{O_{n-2}\, O_{n-3} \ldots O_{n-q-1}}{2\,(n-q)\, O_{q-1}\, O_{q-2} \ldots O_1} M_{q-1}^n \qquad (1 \leq q \leq n-1)$$

and

$$H_q^r = \frac{O_{r-2}\, O_{r-3} \ldots O_{r-q-1}}{2\,(r-q)\, O_{q-1}\, O_{q-2} \ldots O_1} M_{q-1}^r \qquad (1 \leq q \leq r-1)\,\cdot$$

Consequently, in terms of H_q^n, H_q^r the formulae (13), (14) and (15) may be written

(16) $$H_q^n = c_{rnq}\, H_{q+r-n}^r$$

where c_{rqn} are the following constants:

$$c_{rqn} = \frac{\binom{r-1}{n-q}}{\binom{n-1}{q-1}} \cdot \frac{O_{n-2} \ldots O_{r-1}}{O_{q-1} \ldots O_{q+r-n}}\, \frac{O_{q-1}}{O_{q-n+r-1}}, \qquad \text{for} \quad q \geq n-r+1$$

$$c_{rqn} = \frac{O_{n-2}\, O_{n-3} \ldots O_{n-q-1}}{2\,(n-q)\binom{n-1}{q-1} O_{q-1}\, O_{q-2} \ldots O_1}\, O_{q-1} \qquad \text{for} \quad q = n-r$$

$$c_{rqn} = 0 \qquad \text{for} \qquad q < n-r\,.$$

The formula (16) has been given by HERGLOTZ and PETKANTSCHIN (see [1], p. 292); however they do not give the explicit values of the constants c_{rqn}.

Examples.

$$c_{213} = \frac{O_1\, O_0}{2 \cdot 2} = \frac{2 \cdot 2\pi}{4} = \pi, \qquad c_{223} = \frac{O_1\, O_1}{2 O_1\, 2} = \frac{\pi}{2}\,\cdot$$

FACULTY OF SCIENCES
UNIVERSITY OF BUENOS AIRES
BUENOS AIRES, ARGENTINA

(Manuscript received January 8, 1957)

BIBLIOGRAPHY

[1] B. PETKANTSCHIN : *Zusammenhaenge zwischen den Dichten der Linearen Unterraeume im n-dimensionalen Raum*, (Integralgeometrie 6), Abhandlungen aus dem Math. Sem. Hansischen Univ., 11, 249–310, (1936).

[2] L. A. SANTALÒ : *Sur la mesure des espaces linéaires qui coupent un corps convexe et problèmes qui s'y rattachent*, Colloque sur les questions de réalité en Géométrie, 177–190, Liège, (1955).

ÖZET

E^n, n boyutlu öklidyen uzay ve L^r de bunun lineer bir alt-uzayı olsun. L^r nin ihtiva ettiği K^r gibi bir konveks cisim, E^n uzayında, *yassı* bir konveks cisim gibi düşünülebilir. K^r cismi, L^r içinde tetkik edilecek olursa, (1) formülü ile tarif edilen

$$M_r^n \, (0 \leq q \leq r-1)$$

ortalama eğriliklerini; E^n nin yassı bir konveks cismi gibi telâkki olunursa,

$$M_q^n \, (0 \leq q \leq n-1)$$

ortalama eğriliklerini haizdir. Bu ortalama eğrilikler, K^r cismine, ε uzaklığında çizilen $K^r(\varepsilon)$ dış paralel konveks cisminin $M_q^n(\varepsilon)$ ortalama eğriliklerinin, ε u sıfıra yaklaştırmak suretiyle bulunan, limitleri olarak tarif edilirler.

Bu makalenin gayesi, bahis konusu M_q^r ve M_q^n ortalama eğrilikleri arasındaki bağıntıyı ifade eden (13), (14), (15) formüllerini ispat etmektir. Bunların neticesi olarak da HERGLOTZ ve PETKANTSCHIN in bir formülü tamamlanmaktadır –formül (16)–.